

On the Use of Computers in Teaching Precalculus and Calculus

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April 19, 1991

*Statement read at the Tercer Congreso Interuniversitario
de Educación Matemática*

I wish to thank the organizers of this Conference for the opportunity given to me to speak here about so important an issue.

I must confess that, since English is not my mother tongue, I am afraid that I shall not be able to express myself in the best chosen words. I apologize for this.

I am Roumanian by place of birth and by education. I had the good fortune to spend my childhood in a civilized and democratic country, which was not poor by the standards of those times.

I began to attend school just before the outbreak of the Second World War. I received the diploma equivalent to the M.S. in Mathematics after nineteen years, and the Ph.D. in Mathematics after thirty years.

Although Roumania passed through a devastating war, during which she shared with Poland the historical honor of having fought both against Nazism and against Communism, albeit under different circumstances, her almost complete destruction occurred only after the war, under communist occupation.

The culture of a people, however, is not easy to destroy, just as it is not easy to build up.

Let me quote from the Report of the National Commission on Excellence in Education, created on August 26, 1981, by the Secretary of Education, T. H. Bell, and published in the *Notices of the AMS*, October 1983:

Our Nation is at risk. Our once unchallenged preeminence in commerce, industry, science and technological innovation is being overtaken by competitors throughout the world. This report is concerned with only one of the many causes and dimensions of the problem, but it is the one that undergirds American prosperity, security and civility. We report to the American people that while we can take justifiable pride in what our schools and colleges have historically accomplished and contributed to the United States and the well-being of its people, the educational foundations of our society are presently being eroded by a rising tide of mediocrity that threatens our very future as a Nation and a people. What was unimaginable a generation ago has begun to occur—others are matching and surpassing our educational attainments.

If an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war.

I know what this means. I witnessed, in Roumania, the destruction of its School and Culture, among other things, at the hands of a foreign occupying power.

I had the good fortune to have excellent professors, from the first elementary grade to the last year at the university.

When I attended it, during the first ten grades, Roumanian school was still patterned after the French model. May I add that already in the first half of the nineteenth century Legendre's *Geometry*, translated into Roumanian, was used in high school.

As a student, beginning in elementary school, I had to understand that the teaching and learning of mathematics had to be based on reason; we had to give the justification for the computations we made. We were taught to make computations as a tool for solving problems, which had to be based on logical thinking. We were not bored with meaningless numerical computations. We had to explain the reasons for the computations we made in solving a problem.

Beginning in the sixth grade we had to do geometrical proofs. I had six years of geometry in high school. There are people in the mathematical profession today who discard Euclidean geometry as obsolete. This is a fundamental mistake. For more than two thousand years the teaching and learning of Euclidean geometry shaped Western Civilization; its appeal was so strong that a philosopher such as Spinoza wrote *Ethica Ordine Geometrico Demonstrata*. Kepler had a deep knowledge of Euclid and Apollonius. Newton mastered the same art, framing the Calculus in geometric terms. What is forgotten today is that Calculus—about which there is now so much “turmoil,” to use the word introduced by Halmos into the dispute—is the Calculus of Euclidean space, not the Calculus of paper or of the computer. Let me quote from the excellent book by Ernst Snapper and Robert J. Troyer, *Metric Affine Geometry*:

It is unfortunate that solid geometry has fallen into disrepute in many high school circles. The reason is twofold. First, for many years, solid geometry meant plugging numbers into unproven mensuration formulas. Of course such activity is neither geometry nor algebra. It is, at best, arithmetic of the deadliest kind.

Let me now quote from the great book by Roger Penrose, *The Emperor's New Mind*. In Chapter 5 he classifies the physical sciences into three categories: Superb, Useful, and Tentative. Of course, he does not make the mistake of confounding mathematical reality with physical reality. Theories are abstract models, which sometimes create a mathematical reality for the concrete physical reality.

The most ancient of the SUPERB theories is the Euclidean Geometry that we learn something of at school. The ancients may not have regarded it as a physical theory at all, but that is indeed what it was: a sublime and superbly accurate theory of physical space and of the geometry of rigid bodies. Why do I refer to Euclidean geometry as a physical theory rather than a branch of mathematics?

Ironically, one of the clearest reasons for taking that view is that we now know that Euclidean geometry is not entirely accurate as a description of the physical space that we actually inhabit!

In the tenth grade I had one full year of trigonometry. We had to know all the basic formulas with proofs, and to apply them to the solution of geometrical problems, trigonometric equations, and identities. In the same year I had to learn the exponential and logarithmic functions. We had six hours of mathematics each week.

Graphing was not emphasized at all, and rightly so: without Calculus, it is only a chapter in the history of drawing.

Let me dwell more deeply on this problem. I am astonished at the stubborn insistence with which graphing is required of college students at the precalculus level. As it is done, it is not mathematics; it is drilling in drawing. This drilling has two aspects.

1. The first is pencil-on-paper graphing. We can read, in one of the precalculus textbooks:

you should be able to sketch any of them as quickly as you can write your name.

In the same textbook we find the exercise:

47. Sketch the graph of the equation

$$y = (1 - 2x) \sin x.$$

In this respect, let me quote Peter D. Lax (*UME Trends*, Vol. 2, No. 2):

Among our many educational crimes, the one that bears the greatest responsibility for the shortage of young Americans choosing mathematics, physics or engineering as careers may very well be the wretched way we teach calculus.

Above we have an example of an educational crime. At the precalculus level, students are not given the tools with which to solve the above exercise. The normal and sound reaction of any student is to throw away a textbook that requires him or her to do something far above the level of explanation given in it.

2. The same thing happens with computer graphing at this level. I have the program **Derive** for sketching the graph of a cubic equation

$$y = x^3 + tx^2 + ax + \cdots.$$

The student is taught how to push the buttons in order to display, on the screen, the graph of this function. The textbook which the **Derive** program is supposed to support, however, does not tell how to locate the points of minimum and maximum, the inflection points, or the intervals of monotonicity and of concavity or convexity; all that counts is to push the buttons and to make a nice picture appear on the screen.

Let me quote from E. G. Effros (*Notices of the AMS*, Vol. 37, No. 5):

For many of us, the weakest component of the *Everybody Counts* program is the great stress placed on calculators and computers. Contrary to the claim in the *Standards* that there is no evidence to suggest that the availability of calculators makes students dependent on them for simple calculations, many of our students have already become calculator idiots. At UCLA we are now witnessing the appearance of students who are so hooked on their calculators that they cannot multiply 9 by 7 in their heads.

But this misses the point that we are not concerned with whether or not our students can find subtle factorizations of fifth-degree polynomials. Too many of our calculus students cannot factor quadratics! When you teach calculus, the students must have the simple algebraic manipulations in their heads and not in their calculators.

Let me add that I consider the use of the word “manipulation,” when applied to a mental process, not the best choice.

In the eleventh grade I had to learn combinatorics, permutations, determinants of order n , and the general theory of systems of m linear equations in n unknowns; Cramer’s theorem and Rouché’s theorem; and also differential calculus for functions of one real variable. We had to learn the sketching of graphs of functions systematically and on a logical basis. I had to learn the two formulas that allow one to find the oblique asymptotes. These formulas are not given in the mammoth textbooks on Calculus here, for instance.

In the final, twelfth grade, I had integral calculus, analytic geometry, and the theory of polynomials, with proofs.

I shall finish the list by saying that the first semester at the university began with the Cantor–Méray construction of the real number system.

I never used computers, or the abacus, which is their original form, in order to learn or understand mathematics.

Why this list of recollections? Because, beginning with the Bell report issued eight years ago, and continuing up to yesterday’s speech of President Bush, official persons, as well as distinguished professors and mathematicians who care for the fate of mathematical education in the United States, all speak of the dismal situation of mathematical education in this country; and because some of the solutions proposed would deal the final blow to mathematical education here.

Let me quote from Bruno Snell, *The Discovery of the Mind*:

Greek is the only language which allows us to trace the true relation between speech and the rise of science; for in no other tongue did the concepts of science grow straight from the body of the language. In Greece, and only in Greece, did theoretic thought emerge without outside influence, and nowhere else was there an autochthonous formation of scientific terms. All other languages are derivative; they have borrowed or translated or got their terms by some other devious route from the Greeks. And it was only with the help of the unique achievement of the Greeks that the other societies were able to progress beyond their own pace of conceptual development.

In Greece, the verbal—and that is to say: the intellectual—seeds of scientific language are of a very ancient date.

This passage shows very clearly that theoretic (that is, scientific) thought has a verbal character. This belongs to Mathematics in the highest degree. Let me quote from P. Halmos (*Focus*, Vol. 10, No. 6):

And, incidentally, and strongly and relevantly, the same sort of ignorance is visible in other subjects, notably in the use of language. When I require that the solutions of problems be communicated to me not just by scribbled equations but by sentences that begin with a capital letter, and end with a period, and have a reasonable sprinkling of commas between, my students groan and tremble—they don't believe me, they think I am unreasonable, and they simply don't (can't) do it.

Let me quote again from Bruno Snell:

The citadel (Thebes) had been ruled by Cadmus, who brought with him from Phoenicia the art of writing, the source and foundation of all civilized life.

Writing is not the same thing as pushing buttons. Writing is the material recording of mental processes. If the mental processes occur, and they should whenever we are awake—moreover, learning is impossible without them—then, in order to record them, it is enough to have paper and pencil; perhaps a typewriter. To use the buttons of a computer in order to call into operation a program which displays the solution of a problem already solved by the specialists of the company that sold the program, and to pretend that the student solved the problem—moreover, that he or she understood the solution—is a pure fallacy.

Moreover, Computer Mathematics is definitely different from the Mathematics of the Continuum that underlies Calculus. This is very well explained in the book of Roger Penrose which I cited and quoted above. Let me quote again, first, about the physiology of the brain:

The regions of the cerebral cortex just referred to (the visual, auditory, olfactory, somatosensory, and motor) are called primary, since they are the ones most directly concerned with the input and the output of the brain. Near to these primary regions are the secondary regions of the cerebral cortex, which are concerned with a more subtle and complex level of abstraction. . . The remaining regions of the cerebral cortex are referred to as tertiary (or the association cortex). It is largely in these tertiary regions that the most abstract and sophisticated activity of the brain is carried out. It is here—in conjunction, to some extent, with the periphery—that the information from various different sensory regions is intertwined and analysed in a very complex way, memories are laid down, pictures of the outside world are constructed, general plans are conceived and evaluated, and speech is understood or formulated. Speech is particularly interesting, because it is normally thought of as something very specific to human intelligence.

It is clear from this text that the teaching and learning of mathematics should be accompanied by the shifting of information from the primary cortex to the tertiary cortex. When I was a student, my teachers used to say: “to see with the eyes of the mind.”

Looking at the screen of the computer means remaining in the realm of the primary cortex. We should educate our students to operate with the tertiary cortex. This is not easy, and computer programs such as **Derive**, and others, will not help: mathematics happens at the level of speech, not at the level of the primary cortex.

Computer Mathematics is algorithmic, discrete, limited in space and in time. Classical mathematics refers to the Platonic world of concepts and ideas. When doing mathematics we assign a meaning to the terms; we are speaking about a real world of mathematical objects. Computer mathematics is either meaningless, or a fragment of formal mathematics, or it displays a physical object purported to be the mathematical solution.

The book of Penrose also contains a short discussion of the non-verbality of creative thought, as confessed by Einstein, Galton, Hadamard, and Penrose himself. Let me quote again:

I quote these examples because they very much accord with my own thought-modes. Almost all my mathematical thinking is done visually and in terms of non-verbal concepts,...

Of course, we are dealing here with visualization at the level of imagination, because—and I quote again—

Also, the difficulty that these thinkers have had with translating their thoughts into words is something that I frequently experience myself. Often the reason is that there simply are not the words available to express the concepts that are required.

This shows that mathematical understanding and creation take place, perhaps, at the level of what I might venture to call the quaternary cortex, where inspiration, insight, and originality occur.

Let me also reproduce “the non-algorithmic/algorithmic distinction” given by Penrose:

Consciousness needed	Consciousness not needed
“common sense”	“automatic”
“judgement of truth”	“following rules mindlessly”
“understanding”	“programmed”
“artistic appraisal”	“algorithmic”

The section “Contact with Plato’s world” tells more clearly what kind of visualization is involved in the emergence of mathematical concepts.

The computer manufacturing companies might offer just one more unit: one in which the student inserts the textbook; the unit reads off the problems and displays the solutions. The students would then be relieved even of the task of pressing the buttons.

The situation established by the Bell report was not created by the fact that, until then, no computers were used to teach mathematics. It will not be improved by the introduction of computers into teaching. Let me quote from the Bell report again:

Our goal must be to develop the talents of all to their fullest. Attaining that goal requires that we expect and assist all students to work to the limits of their capabilities. We should expect schools to have genuinely high standards rather than minimum ones, and parents to support and encourage their children to make the most of their talents and abilities.

And later:

A Word to Students:

You forfeit your chance for life at its fullest when you withhold your best effort in learning.

Let me end by referring to the excellent article by P. Hilton and Jean Pedersen (“Turmoil in Teaching: a Response to Paul Halmos,” *UME Trends*, Vol. 3, No. 1), from which I quote:

We would go further and assert that a fundamental problem in this country is that training is confused with education and substituted for it—and training that does not lead to any real understanding is always bad education.

May I add that computer-assisted teaching of Calculus, or Precalculus, as done by many computer-algebra programs, is not even training, a word whose meaning has a practical connotation: it is useless drilling.

Drilling should be reserved for military service; training should fall as a task upon companies. The aim of the University is to educate the mind, as its very name suggests. I quote from Dirk J. Struik, *A Concise History of Mathematics*:

Modern mathematics was born in this atmosphere of Ionian rationalism, the mathematics which not only asked the Oriental question ‘how?’, but also the modern scientific question ‘why?’.

And later:

The early Greek study of mathematics had one main goal, the understanding of man’s place in the universe according to a rational scheme. Mathematics helped to find order in chaos, to arrange ideas in logical chains, to find fundamental principles.

The ideas I have expressed above should not be construed as a rejection of computers on my part; that would be silly. What I maintain is that mathematics is transmitted at the verbal level of perception. Computers, besides the amazing practical uses they have, can be used profitably, at the theoretical level, only by those who already know mathematics. Let me give some negative examples of an interrogative character.

1. The derivation of the formula for the distance between two lines in space. How can one prove, on a computer, that it is given by the common perpendicular, and also that this perpendicular is unique?
2. Schwarz's theorem on the equality of the mixed partial derivatives. How can one produce, on the computer, the graph of the classical counterexample in order to show that they are not equal?
3. How can one show, on the computer, that the vector product of two vectors is perpendicular to the two factors?

There is another objection of principle: if the computer gives an answer and the student does not have the preparation to confirm or to reject it, what is the scientific basis of its acceptance? It is clear that we have here a vicious circle which eliminates computers from the business of teaching classical mathematics (not from teaching students to work with computers, which is a totally different thing).

Let me quote from Descartes:

The first precept was never to accept anything for true which I did not clearly know to be such; that is to say, carefully to avoid precipitancy and prejudice, and to comprise nothing more in my judgement than what was presented to my mind so clearly and distinctly as to exclude all ground of doubt.

One of the methods of solving a difficult problem is to replace it by an easier one. Since *calculus* means “stone,” replace the teaching and understanding of Differential and Integral Calculus by playing with stones—the buttons of a calculator.

Thank you very much for your attention.