Physical Applications of the Stevens' Classification of Random Variables

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Abstract

Restrictions on admissible changes of scale are often overlooked in science education. Ironically, this topic receives far more attention in philosophy, psychology, and sociology texts than in the natural sciences. In fact, it was the Harvard psychologist S. S. Stevens [4] who introduced the now-classic taxonomy of measurement scales, classifying variables according to the transformations permitted under a change of scale. This paper presents a brief summary of Stevens' taxonomy and illustrates each category with examples drawn from physics.

1 Introduction

Let Ω be the set of all outcomes of a random experiment and let $X(\omega)$ be a random variable defined for each outcome $\omega \in \Omega$. If g is a real-valued function, then the composition $Y = g(X(\omega))$ defines a new random variable Y often referred to as the *transformed* random variable.

$$\begin{tikzcd} & \Omega \arrow[dl, "X"'] \arrow[dr, "g\circ X"] & \\ \mathbb{R} \arrow[rr, "g"'] & & \mathbb{R} \end{tikzcd}$$

Such transformations have important mathematical and physical applications. Common statistical applications include standardization, grouping of data, changing the form of a distribution, and changes of frame and scale. For example, if X is a random variable with mean μ and variance σ, then $g(x) = (x - \mu)/\sigma$ transforms X to a "standardized" random variable Z with mean 0 and variance 1. Geometrically, $Z(\omega)$ is just the signed distance between $X(\omega)$ and the mean μ measured in standard deviations. If $g(t) = (t - \mu)^2$, then $Y(\omega) = g(X(\omega))$ is a random variable whose mean EY is just the variance of X. If X is a continuous random variable and g a step function approximating the identity function, then the transformed variable Y will be a discrete variable with parameters and characteristics not too different from those of X, a fact useful in the grouping of experimental data and its depiction by, say, a histogram. In Physics, g can represent a Galilean or Lorentzian change of frame. Stevens' classification [4], which we now examine, highlights certain special forms of the function g.

2 Stevens' Classification of Random Variables

In Stevens' classification [4] a variable is called *ratio*, *interval*, *ordinal* or *nominal* according to the class of transformations permitted under a change of scale. The classes of admissible transformations are summarized in Table 1.

Table 1: Classes of Variables

Variable Class	**Transformation Type**	**Admissible Function**
Ratio	Linear	$g(t) = ct$, $c > 0$
Interval	Affine	$g(t) = ct + b$, $c > 0$
Ordinal	Order preserving	$t < t' \implies g(t) < g(t')$
Nominal	Injective	$t \neq t' \implies g(t) \neq g(t')$

3 Examples

3.1 Ratio Variables

In a well-known conversation with Heisenberg cited in [1], Albert Einstein stated that

> "...On principle it is quite wrong to try founding a theory on observable magnitudes alone. In reality the very opposite happens. It is theory which decides what we can observe."

Einstein was very well aware that the axioms of a given physical theory can, and frequently do, impose restrictions on the class of functions that can be used to make changes of scale. This, in turn, imposes limits on the statistical quantities that can be "meaningfully" computed. A good example of this is the quantity length. That length is a ratio variable follows from the following theorem.

Claim 1. *Let $X : \Omega \to \mathbb{R}$ be a random variable taking non-negative values, and let $\oplus$ be an operation on the underlying physical objects such that*

$$X(B \oplus B') = X(B) + X(B')$$

for all bodies B, B'. If a change of scale $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$g[X(B \oplus B')] = g[X(B)] + g[X(B')]$$

and g is continuous from the right at the origin, then $g(x) = cx$ with $c > 0$. Hence X is a ratio random variable in Stevens' taxonomy.

Proof. From the two displayed equations we have

$$g[X(B) + X(B')] = g[X(B)] + g[X(B')].$$

If this is to hold for all B and B', then the function g must satisfy

$$g(x + y) = g(x) + g(y) \tag{1}$$

for all $x, y \geq 0$. The additivity condition (1) is not quite enough to ensure the linearity of g. However, right-continuity of g at the origin is sufficient to prove that g must be linear: $g(x) = cx$. Moreover, since X is non-negative, $c = g(1) > 0$. This proves that length is a ratio variable in Stevens' taxonomy. □

Example 1

In non-relativistic physics the total length of a concatenation of two objects is the sum of their lengths. Moreover, if the lengths of a sequence of bodies tend to zero in one scale, then they tend to zero in any scale. This additional physical axiom guarantees the right-continuity of the change-of-scale function g. It follows that length is a ratio variable. Thus, for example, to change yards to feet we use the linear transformation $L' = 3L$. Since linear functions preserve the arithmetic mean, under a change of length scale from yards to feet an average length in yards will transform correctly to an average length in feet. Thus, the concept of average length is scale-independent and statistically meaningful.

Remark. *Ratio variables arise whenever objects can be concatenated in such a way that the hypothesis of Claim 1 holds. Examples of such variables in physics are time intervals, angle, mass, etc. However, variables such as position on a time scale and position on a length scale are not ratio variables since their admissible class of changes of scale is larger.*

Example 2

Truesdell [6] shows, using Carnot's general axiom (universality of efficiency for reversible engines) together with the first law and the Clausius–Duhem inequality, that one particular temperature scale becomes distinguished: the absolute temperature Θ. It is unique up to a positive multiplicative constant and is therefore a ratio-scale quantity. This is the basis for the modern Kelvin scale, which is ratio.

Remark. *Example 2 shows that the concatenation condition in Claim 1 is not a necessary condition for the linearity of a scale change. Two bodies, each at temperature* $10\,\mathrm{K}$*, do not combine to produce a concatenation with temperature* $20\,\mathrm{K}$*. Yet, as Example 2 indicates, the Kelvin temperature is a ratio variable.*

3.2 Interval Variables

Recall that a random variable is called *interval* if it admits only affine changes of scale $g(t) = at + b$, $a > 0$.

Claim 2. *If $Z(B) = X(B) - X(B_0)$ is a ratio variable (a length), where B_0 is a fixed point, then X is an interval random variable.*

Proof. Let $Y(B) = g \circ X(B)$ be a change of coordinates. Since Z is a ratio variable,

$$Y(B) - Y(B_0) = a\big(X(B) - X(B_0)\big), \quad a > 0.$$

It follows that the coordinate change g is affine. □

Example 3

The time of occurrence of an event is an interval variable whereas the time elapsed during some process is a ratio variable.

Example 4

In an affine space, an interval change of scale $g : x \mapsto ax + b$ corresponds to a change of unit of length $x \mapsto ax$ followed by a change of origin $x \mapsto x + b$.

Example 5

Celsius and Fahrenheit are interval scales: they are affine transformations of Kelvin, $T' = aT + b$ $(a > 0)$. Under a change of scale, temperature differences in Celsius/Fahrenheit are preserved, but ratios are not (hence "20°C is not twice as hot as 10°C").

3.3 Ordinal Variables

Recall that a random variable is called *ordinal* if it admits only strictly increasing changes of scale ($t < t' \implies g(t) < g(t')$).

Claim 3. *Let X be a random variable defined on a sample space Ω on which an order $\prec$ is defined. If*

$$B \prec B' \implies X(B) < X(B'),$$

then X is ordinal.

Proof. Let $Y(B) = g(X(B))$ be a change of scale. Then

$$B \prec B' \Leftrightarrow Y(B) < Y(B')$$

and so

$$X(B) < X(B') \Leftrightarrow g(X(B)) < g(X(B')).$$

If this is to hold for all $B, B' \in \Omega$, the change-of-scale mapping g must be a strictly increasing function. □

Example 6

In the empirical thermometry of the 19th century, the only restriction on temperature was the assumption that the set of all hotnesses of a body comprise a one-dimensional manifold with an intrinsic ordering $\prec$ ("hotter than") and that a change of temperature scale must preserve this order [6, 5]. Thus, by the preceding claim, changes of temperature must be at least ordinal. However, Example 2 shows that the modern Kelvin temperature scale is actually ratio.

See [3, 2] for information concerning the evolution of the thermodynamic temperature scales.

3.4 Nominal Variables

A random variable is nominal if it admits only injective changes of scale:

$$t \neq t' \implies g(t) \neq g(t').$$

Example 7

Consider a variable such as particle type:

$$X = \begin{cases} 1 & \text{if type} = \Omega^- \\ 2 & \text{if type} = K \text{ meson} \\ 3 & \text{if type} = \Sigma \text{ hyperon} \end{cases}$$

in which no particular order relation is specified on the set of particles. The variable X is now merely an indicator of category and the values of X are completely arbitrary. A change of scale such as $g(1) = 2$, $g(2) = 1$ (which transposes Ω^- and K) is perfectly admissible, as would any arbitrary injection. Thus, in Stevens' taxonomy, X would be nominal.

4 Concluding Remarks

Stevens' classification is important in determining the physical meaningfulness of a parameter. For example, the arithmetic mean is not preserved under nonlinear transformations and so, for instance, the concept of a global mean surface temperature would be meaningless in the empirical thermometry of the 19th century. At least an interval or ratio temperature scale is required. The effects of a Celsius doubling of temperature would in general be different from a doubling in degrees Fahrenheit. Doubling is only meaningful in the ratio Kelvin scale.

References

[1] Timothy Ferris. *The Red Limit: The Search for the Edge of the Universe*. William Morrow, 1977.

[2] Jiří J. Mareš. "Do We Know What the Temperature Is?" In: *arXiv preprint* arXiv:1604.05563 (2016). Also published in Journal of Thermal Analysis and Calorimetry, vol. 120, pp. 223–230 (2015). DOI: 10.48550/arXiv.1604.05563. URL: https://arxiv.org/abs/1604.05563.

[3] Andrea Peruzzi. *Time Evolution of the Thermodynamic Temperature Scale*. Presentation at the CCT Meeting Online 2020 (AP-CCT20-56), Consultative Committee for Thermometry (CCT), Bureau International des Poids et Mesures (BIPM). BIPM, Feb. 2020. URL: https://www.bipm.org/documents/20126/54468330/Talk_CCT_Meeting_Online_2020_AP-CCT20-56.pdf.

[4] S. S. Stevens. "On the Theory of Scales of Measurement". In: *Science* 103 (1946), pp. 677–680.

[5] C. Truesdell and S. Bharatha. *Classical Thermodynamics as a Theory of Heat Engines*. Springer Verlag, 1977.

[6] Clifford Truesdell. *Rational Thermodynamics*. 2nd ed. Second edition, corrected and enlarged. New York: Springer-Verlag, 1984. DOI: 10.1007/978-1-4612-5206-1.