Commutative Diagrams and the Construction of Musical Scales

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Introduction

This article is intended as a minimal introduction to some basic musical ideas which fall within the intersection of mathematics and music. We discuss the science underlying the construction of Pythagorean and other tunings, giving mathematically clear definitions of the necessary musical terms. Careful distinction is made between the frequential and perceptual measures of pitch. The admissible changes of scale for each of these measures are derived. It is shown that commutative diagrams can provide a useful tool for the construction and analysis of scales. All needed musical notions are explained. Suggestions for student exercises are provided.

The Physics of Music

The Nature of Sound

Sounds are audible vibrations of the air. Some sounds, because of the structure and frequency of their waveform, have a perceived highness which is called *pitch*. Less structured sounds, such as white noise, rustling leaves, and the percussive sounds of cymbals which have complex, non-periodic waveforms, are said to be *unpitched*. Here we will confine our attention to sounds to which a pitch can be ascribed.

Overtone Series

Hermann von Helmholtz's exhaustive 1875 text [1] carefully analyzes the physiology behind the perception of sound showing how the quality of a musical sound depends on its overtone structure. When an object vibrates at some pitch, a set of distinct frequencies can be detected, the lowest of which is called the *fundamental*. The higher frequencies, called *overtones*, are typically *harmonics*, meaning that they are integer multiples of the fundamental frequency. In real-world musical instruments (see, for example, [2], [3] or [4]), overtones may include slight deviations or non-harmonic components due to material properties or instrument design. Henceforth, we will ignore this possibility and assume that, whenever a note X with fundamental frequency f_{X} is played, detectable frequencies will be terms of the arithmetic sequence $[\mathrm{X}] = (f_{\mathrm{X}}, 2f_{\mathrm{X}}, 3f_{\mathrm{X}}, \ldots)$. This sequence is called the *harmonic* or *overtone* series. Overtones determine what musicians refer to as the *timbre* of the note. Even in the case when overtones are harmonics, the relative amplitudes within an overtone series will of course be different for different instruments, and so the same notes will be perceived differently.

Octave Series

For any note X, the note 2X, with frequency twice that of X, is said to be an *octave above* X and the note $\frac{1}{2}X$, with half the frequency of X, one *octave below*. The *octave series* of X is the geometric sequence of frequencies $(2^n f_{\mathrm{X}})_{n=-\infty}^{\infty}$. Two notes are said to be *octave equivalent* if their frequencies are terms in the same octave series. In musical set theory, the *pitch class* of a note X is just the set of all notes which are octave equivalent to X. In practice, any note in the pitch class of X can act as class representative.

For example, the pitch class A of the standard orchestral note A4 of frequency $f_{A4} = 440\,\mathrm{Hz}$ is the set

$$\mathrm{A} = \{\ldots, \mathrm{A}(-1), \mathrm{A0}, \mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{A4}, \mathrm{A5}, \ldots\},$$

where $f_{A(i+1)} = 2f_{Ai}$ for all i. The notes A(-1) = 13.75 Hz and below fall outside the range of human hearing (typically from 20 Hz to 20 000 Hz).

Properties of Octave and Overtone Series

It is easy to see that the terms of the overtone series of octave equivalent notes form a sequence of nested sets:

$$\cdots [4\mathrm{X}] \subset [2\mathrm{X}] \subset [\mathrm{X}] \subset [\tfrac{1}{2}\mathrm{X}] \subset [\tfrac{1}{4}\mathrm{X}] \subset \cdots$$

and that

$$[n\mathrm{X}] \cap [m\mathrm{X}] = [\max\{n, m\}\mathrm{X}], \text{ whenever } n\mathrm{X} \text{ and } m\mathrm{X} \text{ are octave equivalent.}$$

These properties of overtone series explain why all notes in the pitch class of X have some similarity of timbre. More generally, it turns out that notes whose overtone series have a high degree of intersection combine harmoniously. The highest possible degree of intersection of distinct pitch classes occurs when the notes are an octave apart (in frequency ratio (2:1)), in which case their overtone series share every second harmonic of the lower pitch.

The most important ratio in music after the octave (2:1) is 3:2. For any note X, the note $\frac{3}{2}X$ is said to be one *fifth above* X and $\frac{2}{3}X$ one *fifth below*. Unlike with octaves, the overtone series of consecutive fifths are not nested. Consecutive fifths belong to distinct pitch classes. On the other hand,

$$\begin{aligned}
[\mathrm{X}] \cap [\tfrac{3}{2}\mathrm{X}] &= (\mathrm{X}, 2\mathrm{X}, 3\mathrm{X}, \ldots) \cap (\tfrac{3}{2}\mathrm{X}, 3\mathrm{X}, \tfrac{9}{2}\mathrm{X}, 6\mathrm{X}, \ldots) \\
&= (3\mathrm{X}, 6\mathrm{X}, 9\mathrm{X}, \ldots) \\
&\subseteq [\mathrm{X}].
\end{aligned}$$

That the intersection of these overtone series has a high intersection with that of X suggests that the note $\frac{3}{2}$X will combine "harmoniously" with X. Moreover, since the set difference $[\frac{3}{2}\mathrm{X}] \setminus [\mathrm{X}]$ contains notes not in the overtone series of X, $\frac{3}{2}\mathrm{X}$ will have a musical character distinct from X. This is the essential reason why the linear function[1] $\varphi(x) = \frac{3}{2}x$ plays such an important role in the construction of musical scales.

Building on these ideas, it is found that whenever two notes are in a simple ratio to each other, their overtone sequences will intersect in a harmonious manner. Moreover, the simpler the ratio, the greater the degree of the harmony, a situation analogous to the frequency of conjunction of planets when their periods are in simple ratio to each other.[2]

Measures of Pitch

In music, a change in pitch (up or down) is called an *interval* (not to be confused with an interval on the number line). Changes in pitch are correlated with changes in vibrational frequency.

A finding, commonly attributed to Helmholtz [1] but likely implicit in the work of earlier authors, is that there exists an (approximately) linear relationship between octave measure and perception of a musical interval. Octave measure, which depends on vibrational frequency, is scientifically rigorous, whereas perception is subjective. In what follows, we provide definitions which formalize the connections between these two viewpoints, the frequential and the perceptual.

Suppose that intervals $I_1 = [f_1, f_2]$ and $I_2 = [f_2, f_3]$ are consecutive increases in frequency. The interval $[f_1, f_3]$, representing the total increase in pitch, will be called the *concatenation* of the two intervals and denoted $I_1 \oplus I_2$. Experiment shows that when two increases in pitch "sound the same," their frequency ratios concatenate multiplicatively. That is,

$$\frac{f_3}{f_1} = \frac{f_2}{f_1} \times \frac{f_3}{f_2}. \tag{1}$$

(See Figure 1.) Notice that this relation is invariant under linear changes in the frequency scale so that if all frequencies are scaled to a fundamental frequency f_F then

$$\frac{f_3/f_F}{f_1/f_F} = \frac{f_2/f_F}{f_1/f_F} \times \frac{f_3/f_F}{f_2/f_F}.$$

This experimental finding motivates the following definition which characterizes the difference between perceptual and frequential measures of pitch.

Definition 1. *Let $\mathcal{I}$ be the set of all possible musical intervals. A function $m : \mathcal{I} \to \mathbb{R}$ will be called:*

[1] There are advantages in working with linear functions rather than directly with ratios as is more usual.

[2] Kronecker's theorem on the n-dimensional torus describes the situation when the periods of the planets are not rationally related.

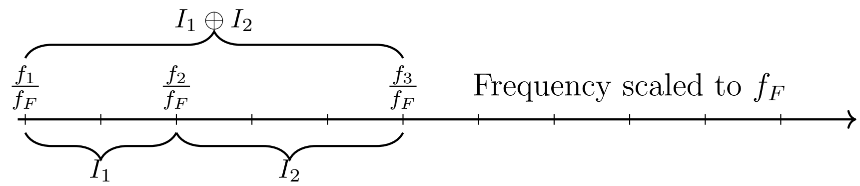

Figure 1: Frequential Concatenation of Musical Intervals

1. *an auditive or perceptual pitch measure if*

$$m(I \oplus I') = m(I) + m(I'). \tag{2}$$

2. *a frequential pitch measure if*

$$m(I \oplus I') = m(I)m(I'). \tag{3}$$

for any intervals I and I'.

We will assume that $\mathcal{I}$ is sufficiently rich in intervals that auditive measures of pitch achieve all real numbers, and that frequential measures assume all positive numbers. To avoid pathological cases, auditive measures of interval pitch will be assumed to satisfy the following continuity axiom:

Axiom 1. *Let $m : \mathcal{I} \to \mathbb{R}$ be an auditive pitch measure, and let $\{I_n\}$ be any sequence of auditive pitch intervals. If $\lim_{n\to\infty} m(I_n) = 0$, then the same holds for all auditive pitch measures.*

Notice that (2) is just the familiar rule for the measure of the union of disjoint intervals. In non-relativistic physics, it might serve as an axiom for the length of the concatenation of mechanical bodies, and is the reason why changes of length scale, such as from feet to yards, are necessarily linear.

Changes of Pitch Interval Measure

Let $m, m' : \mathcal{I} \to \mathbb{R}$ be two interval measures. An increasing function h such that $m'(I) = h(m(I))$ for every interval I will be called a *change of pitch scale.*

Theorem 1. *Equations (2) and (3) and Axiom (1) place the following restrictions on the admissible changes of pitch:*

1. *Auditive to Auditive:* $x \mapsto cx, \quad x \in \mathbb{R}.$

2. *Auditive to Frequential:* $x \mapsto e^{cx}, \quad x \in \mathbb{R}.$

$$
\begin{tikzcd}
\mathbb{R} \arrow[rr, "x\mapsto cx"] & & \mathbb{R} \arrow[dd, "x\mapsto e^{cx}"] \\
& \mathcal{I} \arrow[ul, "m_a"'] \arrow[ur, "m'_a"] \arrow[dl, "m_f"] \arrow[dr, "m'_f"'] & \\
\mathbb{R}^{>} \arrow[uu, "x\mapsto \log_c(x)"] \arrow[rr, shift left, "x\mapsto x^c"] & & \mathbb{R}^{>} \arrow[ll, shift left, "x\mapsto x^{1/c}"]
\end{tikzcd}
$$

Figure 2: Admissible Changes of Pitch Scale.

3. *Frequential to Auditive:* $\quad x \mapsto \log_c x, \quad x > 0.$

4. *Frequential to Frequential:* $\quad x \mapsto x^c, \quad x > 0.$

The four cases are summarized in Figure 2.

Proof. Case 1. Suppose m_a, m'_a are auditive pitch measures and h a change of pitch scale, so that $m'_a(I) = h(m_a(I))$. Using Equation (2) we have

$$
\begin{aligned}
h(m_a(I)) + h(m_a(I')) &= m'_a(I) + m'_a(I') \\
&= m'_a(I \oplus I') \\
&= h(m_a(I \oplus I')) \\
&= h(m_a(I) + m_a(I')).
\end{aligned}
$$

Since $\mathcal{I}$ is sufficiently rich in intervals so that m_a and m'_a achieve all real numbers, then the function $h : \mathbb{R} \to \mathbb{R}$ must satisfy Cauchy's functional equation:

$$h(x + x') = h(x) + h(x'). \tag{4}$$

Moreover, from Axiom (1) it follows easily that h must be continuous at the origin. It is well known [5] that additivity together with right continuity at the origin are sufficient to guarantee that an increasing, additive function $h : \mathbb{R} \to \mathbb{R}$ has the form $h(x) = cx$ for some $c > 0$.

Case 2. Suppose that measure m_a is auditive, measure m_f frequential, and that h is a change of scale from m_a to m_f. Then,

$$
\begin{aligned}
h(m_a(I) + m_a(I')) &= h(m_a(I \oplus I')) \\
&= m_f(I \oplus I') \\
&= m_f(I) \cdot m_f(I') \\
&= h(m_a(I)) \cdot h(m_a(I')).
\end{aligned}
$$

Since m_a achieves all real numbers, then h must satisfy the functional equation

$$h(x + x') = h(x) \cdot h(x'), \quad x, x' \in \mathbb{R}. \tag{5}$$

Since h is increasing and continuous, it is well known that $h(x) = e^{cx}$ for some $c > 0$.

Case 3. Suppose that measure m_a is auditive, measure m_f frequential, and that h is a change of scale from m_f to m_a. Then,

$$\begin{aligned} h(m_f(I) \cdot m_f(I')) &= h(m_f(I \oplus I')) \\ &= m_a(I \oplus I') \\ &= m_a(I) + m_a(I') \\ &= h(m_f(I)) + h(m_f(I')). \end{aligned}$$

Since m_f achieves all positive numbers, then h must satisfy the functional equation

$$h(xx') = h(x) + h(x'),\ x, x' > 0. \tag{6}$$

Since h is increasing and continuous, it is well known that $h(x) = \log_c(x)$.

Case 4. Suppose that m_f and m'_f are frequential measures of pitch and h a change of scale with $m'_f = h(m_f)$. Then, using (3) we have

$$\begin{aligned} h(m_f(I) \cdot m_f(I')) &= h(m_f(I \oplus I')) \\ &= m'_f(I \oplus I') \\ &= m'_f(I) \cdot m'_f(I') \\ &= h(m_f(I)) \cdot h(m_f(I')). \end{aligned}$$

Again, assuming that both measures achieve all non-negative numbers, the function h must satisfy

$$h(xx') = h(x)h(x'), \quad x > 0. \tag{7}$$

which has continuous solutions of the form $k(x) = x^c,\ c > 0$. □

Remark. *Case 4 is not musically interesting. Although, for example, a function such as $k(x) = \sqrt{x}$ preserves the concatenation relation (3), it is difficult to envisage a use for such a scale change.*

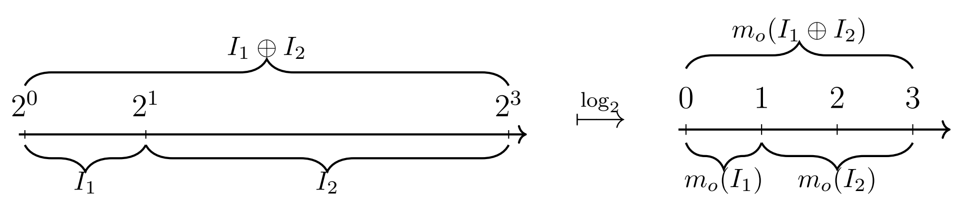

Figure 3: Frequential to Octave Scale Change

Corollary 1. *There exists a linear relationship between octave and perceptual measures.*

$$\begin{tikzcd}
 & \mathcal{I} \arrow[d, "m_f : I \mapsto \frac{f_2}{f_1}"] & \\
 & \mathbb{R}^{>} \arrow[dl, "m_o : x \mapsto \log_2 x"'] \arrow[d, "m_s"'] \arrow[dr, "m_c : x \mapsto 1200 \log_2 x"] & \\
O \arrow[r, "x \mapsto 12x"'] & S \arrow[r, "x \mapsto 100x"'] & C
\end{tikzcd}$$

Figure 4: Scale Changes from Frequency to Octave, Semitone, and Cent Interval Measures

Proof. Restated, the assertion is that the number of octaves in an interval is an auditive measure of pitch. Let $I = [f, f']$ be a frequency interval. In the case that I is an exact number of octaves, by definition of the octave series, $f' = 2^k f$ for some integer k (see Figure 3). Hence, there are $m_o(I) = k = \log_2(f'/f)$ octaves in I. If I is not an exact number of octaves, we merely define the number of octaves in I by $m_o(I) = \log_2(f'/f)$. It follows immediately from Equation (1) that m_o is additive under concatenation. $\square$

By further dividing the octave into, say, 12 equal semitones and each semitone into 100 equal *cents*, we obtain two more auditive measures (see Figure 4):

1. $m_s(I) = 12 \cdot m_o(I)$, *Equal Temperament semitone measure of I.*
2. $m_c(I) = 1200 \cdot m_o(I)$, *Cent measure of I.*

If a note F is taken as origin, the *pitch measure* of a note X with respect to F is defined to be the measure of the interval from F to X. The following measures of note pitch are in common use:

1. Octave measure of X.
$$m_o(\mathrm{X}) = \log_2 \frac{f_{\mathrm{X}}}{f_{\mathrm{F}}}$$
2. Equal Temperament semitone measure of X.
$$m_s(\mathrm{X}) = 12 \log_2 \frac{f_{\mathrm{X}}}{f_{\mathrm{F}}}. \tag{8}$$
3. MIDI number of X.
$$N_{\mathrm{MIDI}}(\mathrm{X}) = 12 \log_2 \frac{f_{\mathrm{X}}}{f_{\mathrm{C}(-1)}}.$$
4. Cent measure of X.
$$m_c(\mathrm{X}) = 1200 \log_2 \frac{f_{\mathrm{X}}}{f_{\mathrm{F}}}.$$

The Musical Instrument Digital Interface (MIDI) number of a note X is the number of semitones relative to the note *quadruple low* C, denoted C(-1), of frequency $f_{C(-1)} = 8.1758\,\mathrm{Hz}$. This note, which is produced by a 64 foot organ pipe[3], falls outside the range of human hearing and is felt rather than heard. The MIDI formula is usually rewritten in terms of $f_{\mathrm{A4}} = 440\,\mathrm{Hz}$ which falls exactly 69 semitones above C(-1) in the modern orchestral scale (see later). Thus

$$N_{\mathrm{MIDI}}(\mathrm{X}) = 12\left(\log_2 \frac{f_{\mathrm{X}}}{f_{\mathrm{A4}}} + \log_2 \frac{f_{\mathrm{A4}}}{f_{\mathrm{C}(-1)}}\right) = 12\log_2\left(\frac{f_{\mathrm{X}}}{f_{\mathrm{A4}}}\right) + 69.$$

In the modern Equal Temperament scale (see later), all MIDI numbers are integers. To allow storage in 8 bits of computer memory, they are restricted to the range 0 to 127.

Musical Scales

In musical theory [2], a *scale* is a finite, strictly increasing sequence of musical notes $X_1, X_2, \ldots, X_k$ which subdivide an octave. The first note of a scale is called the *tonic* or *fundamental*. Since each note corresponds to a specific frequency of the air molecule vibrations produced by an instrument, the overtone series of the notes in a scale should be (to the extent possible) harmoniously related. This is accomplished by requiring that the frequency of each note be (at least approximately) a simple ratio of the first note and consequently of its preceding note. In particular, X_k should be an octave above X_1. Other constraints are desirable for a musical scale. Since the ability to distinguish small differences in pitch is limited, the number of notes should not be very large. To facilitate *modulation* or *change of key* (i.e., starting a tune on a different note), the number of distinct frequency ratios between consecutive notes should be quite small. An arbitrary scale can be expressed as a composition

$$f_{X_1} \xmapsto{g_1} f_{X_2} \xmapsto{g_2} \cdots \xmapsto{g_{k-1}} f_{X_{k-1}} \xmapsto{g_k} f_{X_k}.$$

Noting that a composition of linear functions $g \circ h = g \cdot h$ is multiplicative, the necessary property (3) of frequential measures will be satisfied if all the functions g_i are linear. Harmonic purity will be achieved if the slopes correspond to simple ratios. To illustrate these ideas, we now consider a possible construction of the classical Pythagorean scale.

The Pythagorean C Scale

Pythagoras of Samos (sixth century BC) developed a scale with notes C, D, E, F, G, A, B, 2C ([2],[6]). It is conjectured that his construction was based on a study

[3]There are only two examples in the world. The Boardwalk Hall Auditorium Organ in Atlantic City, New Jersey, and the Sydney Town Hall Grand Organ in Sydney, Australia, both have 64-foot pipes.

Table 1: Pythagorean and Just Intonation C Scales Comparing Ratios to the Fundamental.

Scale	C	D	E	F	G	A	B	C’
Pythagorean	1	9:8	81:64	4:3	3:2	27:16	243:128	2
Just Intonation	1	9:8	5:4	4:3	3:2	5:3	15:8	2

$$\begin{tikzcd} \mathrm{C} \arrow[r, "\varphi"] \arrow[dr, "\tau"'] & \mathrm{G} \arrow[d, "\frac{\varphi}{2}"] \\ & \mathrm{D} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{C} \arrow[r, "\varphi"] \arrow[dr, "\tau"'] & \mathrm{G} \arrow[d, "\frac{\varphi}{2}"] & \\ & \mathrm{D} \arrow[r, "\varphi"] \arrow[dr, "\tau"'] & \mathrm{A} \arrow[d, "\frac{\varphi}{2}"] \\ & & \mathrm{E} \arrow[r, "\varphi"] & \mathrm{B} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{C} \arrow[r, "\varphi"] \arrow[dr, "\tau"'] & \mathrm{G} \arrow[d, "\frac{\varphi}{2}"] & & \\ & \mathrm{D} \arrow[r, "\varphi"] \arrow[dr, "\tau"'] & \mathrm{A} \arrow[d, "\frac{\varphi}{2}"] & \\ & & \mathrm{E} \arrow[r, "\varphi"] \arrow[dr, "\ell"'] & \mathrm{B} \arrow[d, "\ell\circ\varphi^{-1}"] \\ & & & \mathrm{F} \arrow[r, "\varphi"] & 2\mathrm{C} \end{tikzcd}$$

Figure 5: Construction of the Pythagorean C Scale by Commutative Diagram

of the ratios between the lengths of the vibrating strings required to produce each note. Physics informs us that for a fixed string tension, the frequency of a note is inversely proportional to the string length. If the frequencies are scaled so that the fundamental note C is assigned a frequency $f_C = 1$ and the other notes expressed as ratios relative to it, the Pythagorean scale divides the octave from C to 2C into the simple ratios shown in Table 1.

Construction of the Pythagorean C Scale by Commutative Diagram

Let φ be the linear function $\varphi(x) = \frac{3}{2}x$. Since $\varphi(\mathrm{C})$ falls between C and 2C and combines harmoniously with C, we include $\varphi(\mathrm{C})$ in the C-scale. A second application of the function φ takes us outside [C, 2C] and so does not produce a note of the scale. On the other hand, the note one octave lower $\frac{1}{2}\varphi \circ \varphi(\mathrm{C}) = \frac{3^2}{2^3}\mathrm{C}$ does fall within the desired interval and, moreover, has overtone series which combines well musically with that of both C and $\varphi(\mathrm{C})$. Hence, we include this note in the scale. The composition

$$\tau(\mathrm{C}) = \frac{1}{2}\varphi \circ \varphi(\mathrm{C}) = \frac{9}{8}\mathrm{C},$$

will be called a *tone*. A post hoc decision to make τ the step size of our scale now suggests including approximately seven tones between C and 2C, and motivates using labels {C, D, E, F, G, A, B}. It is, therefore, natural to define $\mathrm{D} = \tau(\mathrm{C})$ and similarly, since $\varphi(\mathrm{C}) = \mathrm{C} + \frac{4}{8}\mathrm{C}$, to define $\mathrm{G} = \varphi(\mathrm{C})$. Since φ and $\frac{1}{2}\varphi$ are harmonically good mappings, we repeat this procedure to obtain three additional notes:

$$\mathrm{A} \;= \varphi(\mathrm{D}), \quad \mathrm{E} = \frac{1}{2}\varphi(\mathrm{A}), \quad \mathrm{B} = \varphi(\mathrm{E}).$$

See Figure 5. Since composition of linear functions is commutative $\left(\frac{1}{2}\varphi\right) \circ \varphi = \varphi \circ \left(\frac{1}{2}\varphi\right)$, it is seen from the diagram that $A = \tau(\mathrm{G})$ and $B = \tau(\mathrm{A})$. The uniqueness of prime factorization informs us that no positive power of 2 can equal a power of 3. This makes the note 2C unreachable by repetition of this procedure. To complete the scale, we therefore seek a linear function ℓ and a note F with $\mathrm{F} = \ell(\mathrm{E})$ and $2\mathrm{C} = \varphi(\mathrm{F})$ such that the third diagram in Figure 5 commutes (meaning that all paths from one note to another yield the same total change in pitch). The function ℓ can now be determined from the fact that any path from C to 2C must be a composition equal to twice the identity function. For example, following the path (C, G, D, A, E, B, F, 2C) yields

$$\varphi \circ \ell \circ \varphi^{-1} \circ \varphi \circ \frac{\varphi}{2} \circ \varphi \circ \frac{\varphi}{2} \circ \varphi(\mathrm{C}) = 2\mathrm{C}.$$

It follows that

$$\ell(x) = 2^3 \cdot (\varphi^{-1})^{\circ 5}(x) = \frac{2^8}{3^5}x = \frac{256}{243}x.$$

The function ℓ is called a *diatonic semitone* or *limma* (from the Greek λεί̈μμα, meaning "remnant"). Since composition of linear functions is commutative, the intervals between E and F and between B and 2C are both limmas. We see from Figure 5 that

$$\mathrm{G} = \varphi \circ \tau^{-1} \circ \tau^{-1} \circ \ell^{-1}(\mathrm{F}) = \frac{3}{2} \cdot \frac{8}{9} \cdot \frac{8}{9} \cdot \frac{3^5}{2^8}\mathrm{F} = \frac{3^2}{2^3}\mathrm{F} = \tau(\mathrm{F}).$$

Thus, the interval between F and G is also a tone. In summary, we have derived, using the physics of overtones, two functions ℓ, τ such that the Pythagorean scale of Table 1 is given by the composition (well known to every musician)

$$\mathrm{C} \xmapsto{\tau} \mathrm{D} \xmapsto{\tau} \mathrm{E} \xmapsto{\ell} \mathrm{F} \xmapsto{\tau} \mathrm{G} \xmapsto{\tau} \mathrm{A} \xmapsto{\tau} \mathrm{B} \xmapsto{\ell} 2\mathrm{C}$$

Since the interval from C to G is a fifth we deduce the useful relation $\varphi = \tau^{\circ 3} \circ \ell$.

In music, a 7-note scale such as the Pythagorean is called *diatonic* if it divides the octave into 5 equal tones and 2 equal semitones such that the path length between the two semitones is maximal (see Figure 6).

It is important to emphasize that in Pythagorean tuning the concatenation of two semitones is slightly less than a tone. The linear function p such that $\tau = \ell \circ p \circ \ell$ is called the *Pythagorean comma* and is given by

$$p(x) = \ell^{-1} \circ \tau \circ \ell^{-1}(x) = \frac{3^{12}}{2^{19}}x = \frac{531441}{524288}x \approx 1.0136432647705078125x.$$

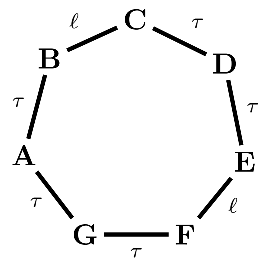

Figure 6: The Interval Structure of a Diatonic Scale (Notes C and 2C are identified)

The composition $a = p \circ \ell$ is called the *chromatic semitone* or *Pythagorean apotome* (from the Greek απεοτομε, meaning "cutting off"). Since $\tau = \ell \circ a$, musicians say (somewhat ambiguously) that a limma is what remains when an apotome has been removed from a tone.

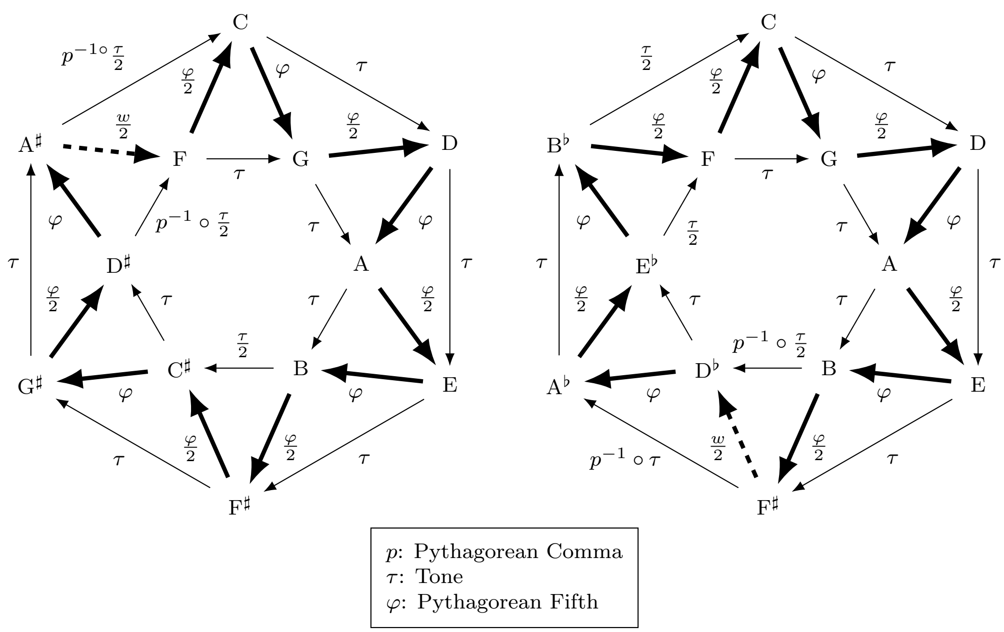

Figure 7: Two Extensions of the Pythagorean Diatonic Scale to 12 Notes Illustrating Hamiltonian Cycles of Fifths (drawn in bold) and Rotation of the Wolf Fifth

Extensions of the Pythagorean Scale to 12 Notes

If music in the Pythagorean scale is to be transposed so as to begin on a note other than C, it is clear that extra notes must be inserted. For example, if the scale is to begin on G preserving the $\tau\tau\ell\tau\tau\tau\ell$ pattern, an extra so called *chromatic* note $F^\sharp = \ell^{-1}(G)$ must be inserted between F and G. Additional changes of key can be accommodated by adding to selected diatonic tonal intervals XY two *chromatic* notes, Y flat and X sharp, defined by $\mathrm{Y}^\flat = \ell(\mathrm{X})$ and $\mathrm{X}^\sharp = \ell^{-1}(\mathrm{Y})$. Recalling that $\tau = \ell \circ a$, this definition expresses an interval XY as the composition of the limma from X to $\mathrm{Y}^\flat$ and the apotome from $\mathrm{Y}^\flat$ to Y

$$\mathrm{X} \xmapsto{\ell} \mathrm{Y}^\flat \xmapsto{a} \mathrm{X}^\sharp \xmapsto{\ell} \mathrm{Y}.$$

When an interval XY is divided in this manner the resulting notes $\mathrm{Y}^\flat$ and $\mathrm{X}^\sharp$ are said to be *enharmonically equivalent*[4]. The diatonic Pythagorean scale can be extended in a multiplicity of ways by including subsets of the chromatic notes. Figure 7 shows two 12 note extensions, generated respectively by the chromatic subsets $\{\mathrm{F}^\sharp, \mathrm{C}^\sharp, \mathrm{G}^\sharp, \mathrm{D}^\sharp, \mathrm{A}^\sharp\}$ and $\{\mathrm{F}^\sharp, \mathrm{B}^\flat, \mathrm{E}^\flat, \mathrm{A}^\flat, \mathrm{D}^\flat\}$. The first graph in Figure 7 contains the Hamiltonian cycle[5]

$$\mathrm{C} \overset{\varphi}{\mapsto} \mathrm{G} \overset{\frac{\varphi}{2}}{\mapsto} \mathrm{D} \overset{\varphi}{\mapsto} \mathrm{A} \overset{\frac{\varphi}{2}}{\mapsto} \mathrm{E} \overset{\varphi}{\mapsto} \mathrm{B} \overset{\frac{\varphi}{2}}{\mapsto} \mathrm{F}^\sharp \overset{\frac{\varphi}{2}}{\mapsto} \mathrm{C}^\sharp \overset{\varphi}{\mapsto} \mathrm{G}^\sharp \overset{\frac{\varphi}{2}}{\mapsto} \mathrm{D}^\sharp \overset{\varphi}{\mapsto} \mathrm{A}^\sharp \overset{\frac{w}{2}}{\mapsto} \mathrm{F} \overset{\frac{\varphi}{2}}{\mapsto} \mathrm{C} \tag{9}$$

Notice that all edges in this cycle, with one exception, are either a fifth or, to keep the notes in the interval C–2C, one half of one fifth. The single exception, from $\mathrm{A}^\sharp$ to F, arises because there is no note in the scale exactly one Pythagorean fifth above $\mathrm{A}^\sharp$. The interval from $\mathrm{A}^\sharp$ to 2F

$$\mathrm{A}^\sharp \xmapsto{\ell} \mathrm{B} \xmapsto{\ell} \mathrm{C} \xmapsto{\tau} \mathrm{D} \xmapsto{\tau} \mathrm{E} \xmapsto{\ell} 2\mathrm{F}$$

is a dissonant interval called the *Procrustean*[6] or *wolf fifth* (due to its wolf-like sound). Notice

$$2\mathrm{F} = \ell^{\circ 3} \circ \tau^{\circ 2}(\mathrm{A}^\sharp) = p^{-1} \circ \varphi(\mathrm{A}^\sharp).$$

Thus, the wolf fifth $w = p^{-1} \circ \varphi$ differs slightly from the usual fifth by an inverse Pythagorean comma. If, say, the note $p^{-1} \circ \varphi(\mathrm{A4})$ one Pythagorean wolf fifth above $\mathrm{A4} = 440\,\mathrm{Hz}$ is played together with the note $\varphi(\mathrm{A4})$ one Pythagorean fifth above A4, a beat frequency of $8.88\,\mathrm{Hz}$, audible as a slow, howling pulsation, can be detected. The musically loathsome presence of this rogue interval cannot be avoided by using a different set of chromatic notes. For example, the alternative choice in Figure 7 merely rotates the dissonant interval.

Composing the pitches along the Hamiltonian cycle (9) yields

$$\frac{1}{2^7} \cdot p^{-1} \circ \varphi^{\circ 12}(\mathrm{C}) = \frac{1}{2^7} \cdot \frac{2^{19}}{3^{12}} \cdot \frac{3^{12}}{2^{12}}(\mathrm{C}) = \mathrm{id}(\mathrm{C}).$$

[4] In older literature, the term *enharmonic* refers to scales with microtonal intervals, such as those found in ancient Greek music. More recently it has come to refer to notes which are written differently but have the same pitch such as $\mathrm{C}^\sharp$ and $\mathrm{D}^\flat$ in the modern orchestral scale.

[5] A closed path containing all vertices of the graph.

[6] To make his guests fit exactly in a bed, the mythical Procrustes had a bed either stretch them or cut their heads off. Analogously, in music, we shorten or stretch intervals to make a diagram commute.

Similarly, it can be verified that the composition of functions around any other directed cycle is the identity map. It is left to the reader to show that commutativity of all cycles in a directed graph whose edges are functions is sufficient for commutativity.

Remark. *A useful geometric property of the commutative diagrams in Figure 7 is that the two vertices at the base of every isosceles triangle determine the important fourth and fifth chords of the scale identified by the apex. Moreover, musicians will recognize that the Hamiltonian subgraph is isomorphic to the well known circle of fifths.*

Extensions of the Pythagorean Scale to 17 Notes

Addition of all chromatic notes extends the diatonic Pythagorean scale to the 17-note scale shown in Table 2 and illustrated by a commutative diagram in Figure 10. This scale contains a Hamiltonian path of fifths of length 16 (Figure 9) and has interval structure

$$\begin{aligned}&1 \xmapsto{\ell} 2 \xmapsto{p} 3 \xmapsto{\ell} 4 \xmapsto{\ell} 5 \xmapsto{p} 6 \xmapsto{\ell} 7 \xmapsto{\ell} 8 \xmapsto{\ell} 9 \xmapsto{p} 10\\ &\xmapsto{\ell} 11 \xmapsto{\ell} 12 \xmapsto{p} 13 \xmapsto{\ell} 14 \xmapsto{\ell} 15 \xmapsto{p} 16 \xmapsto{\ell} 17 \xmapsto{\ell} 1'\end{aligned}$$

It turns out that this scale is a mode (mathematically a rotation) of the ancient Arabic and Persian Scale shown in Figure 8. Citing the writings of the fourteenth century Persian theorist Abdul Kadir, Hermann von Helmholtz [1] deduced that this scale was obtained using a particular sequence of 16 Pythagorean fifths, reduced to the octave and placed in order. The construction explains the existence of the Hamiltonian path (Figure 9). Even with the addition of all of the chromatic notes

$$\begin{aligned}&1)\ \mathtt{C} - 2)\ \mathtt{D}_\flat - 3)\ \grave{\mathtt{D}}_1 \smile 4)\ \mathtt{D} - 5)\ \mathtt{E}_\flat - 6)\ \grave{\mathtt{E}}_1 \smile\\ &7)\ \mathtt{E} - 8)\ \mathtt{F}' - 9)\ \mathtt{G}_\flat - 10)\ \grave{\mathtt{G}}_1 \smile 11)\ \mathtt{G} - 12)\ \mathtt{A}_\flat -\\ &13)\ \grave{\mathtt{A}}_1 \smile 14)\ \mathtt{A} - 15)\ \mathtt{B}_\flat - 16)\ \grave{\mathtt{B}}_1 - 17)\ \grave{\mathtt{c}}_1 \smile 18)\ \mathtt{c}\end{aligned}$$

Figure 8: Arabic-Persian Scale from Helmholtz [1], where the line $-$ between two tones indicates a Pythagorean Limma and $\smile$ a Pythagorean Comma

to the diatonic scale, the resulting 17-note Pythagorean scale (Table 2) still has no note exactly one tone (9:8) above either $D^\sharp$ or $A^\sharp$ and no fifth (3:2) above $A^\sharp$. The major $\tau\tau\ell\tau\tau\tau\ell$ pattern is possible for keys not beginning with $C^\sharp$, $D^\sharp$, E, $G^\flat$, $F^\sharp$, $G^\sharp$, $A^\sharp$, or B.

Construction of the Other Scales by Composition

It is clear from the discussion that choice of the functions τ and ℓ used in the construction of the Pythagorean tuning is far from unique. Many scales based on

Table 2: A 17-Note Pythagorean Enharmonic C Scale showing Pythagorean Commas (p) and Limmas (ℓ)

Note	Ratio	Cents	Note Above	Fifth Above
C	1:1	0.00	D	G
$D^\flat = \ell(C)$	256:243	90.22	$E^\flat$	$A^\flat$
$C^\sharp = p(D^\flat)$	2187:2048	113.69	$D^\sharp$	$G^\sharp$
$D = \ell(C^\sharp)$	9:8	203.91	E	A
$E^\flat = \ell(D)$	32:27	294.13	F	$B^\flat$
$D^\sharp = p(E^\flat)$	19683:16384	317.60	–	$E^\sharp$
$E = \ell(D^\sharp)$	81:64	407.82	$F^\sharp$	B
$F = \ell(E)$	4:3	498.04	G	C
$G^\flat = \ell(F)$	1024:729	588.27	$A^\flat$	$D^\flat$
$F^\sharp = p(G^\flat)$	729:512	611.73	$G^\sharp$	$C^\sharp$
$G = \ell(F^\sharp)$	3:2	701.96	A	D
$A^\flat = \ell(G)$	128:81	792.18	$B^\flat$	$E^\flat$
$G^\sharp = p(A^\flat)$	6561:4096	815.64	$E^\sharp$	$D^\sharp$
$A = \ell(G^\sharp)$	27:16	905.87	B	E
$B^\flat = \ell(A)$	16:9	996.09	C	F
$A^\sharp = p(B^\flat)$	59049:32768	1019.55	–	–
$B = \ell(A^\sharp)$	243:128	1109.78	2C	$F^\sharp$
$C' = \ell(B)$	2	1200	2D	2G

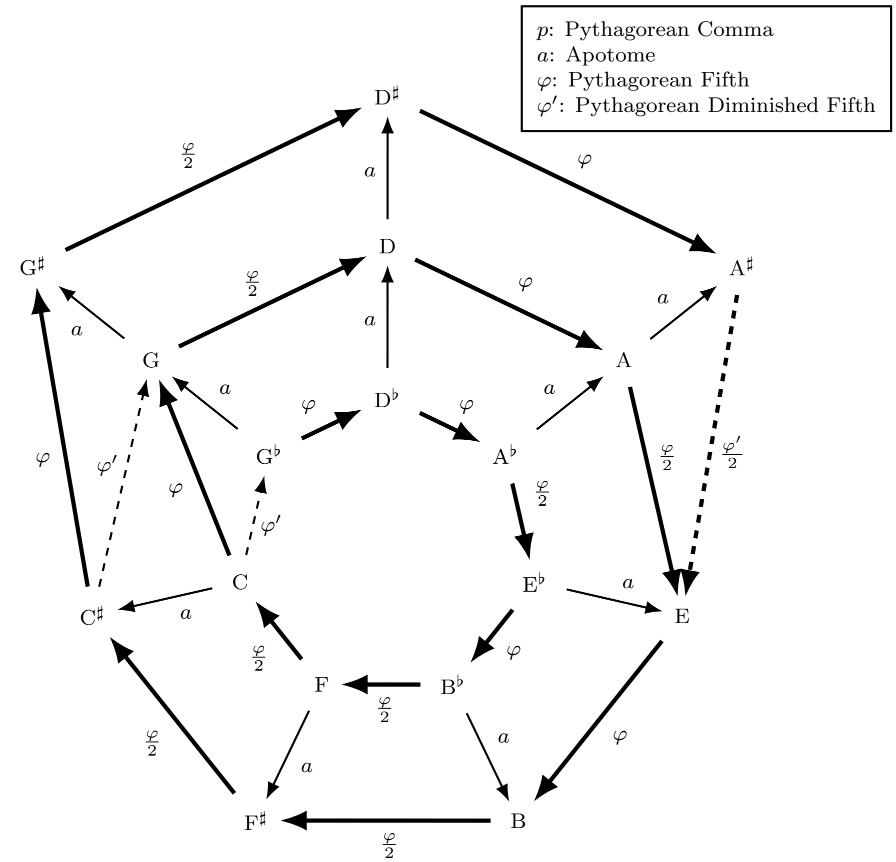

Figure 9: Hamiltonian Path of Fifths Generating a 17-Tone Pythagorean Scale and Illustrating the Location of Apotomes and Diminished Fifths

overtone series can be constructed using a small set of harmonically nice functions. One such example is the scale of Just Intonation (see Table 1), obtained by making each note harmonically very pure with respect to the starting note rather than its preceding note. As a composition of functions, it has the form

$$\mathrm{C} \stackrel{\tau}{\longmapsto} \mathrm{D} \stackrel{\tau'}{\longmapsto} \mathrm{E} \stackrel{\ell'}{\longmapsto} \mathrm{F} \stackrel{\tau}{\longmapsto} \mathrm{G} \stackrel{\tau'}{\longmapsto} \mathrm{A} \stackrel{\tau}{\longmapsto} \mathrm{B} \stackrel{\ell'}{\longmapsto} 2\mathrm{C}$$

where $\tau(x) = \frac{9}{8}x$ is the usual Pythagorean tone, $\tau'(x) = \frac{10}{9}x$ is called a *major tone* and $\ell'(x) = \frac{16}{15}x$ is called a *diatonic semitone*. This scale differs in frequency only slightly from the Pythagorean tuning. However, the fact that three functions τ, τ' and ℓ' are required means that the intervals between notes are not as regular and therefore modulation across keys is harder. Despite this, the scale is harmonically very pure and finds use in vocal and early music. It also appears in many non-Western microtonal musical systems.

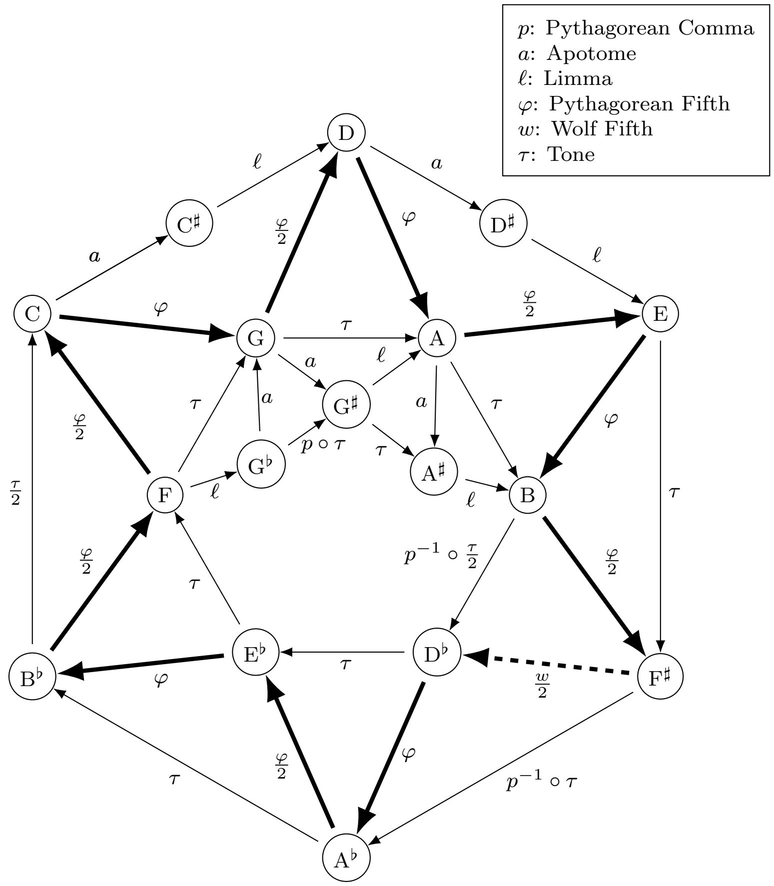

Figure 10: A Commutative Diagram illustrating the 17-Tone Pythagorean Scale

12-Tone Equal Temperament Scales

In order to facilitate changes of key, and eliminate pathologies such as the wolf tone, the *12-tone Equal Temperament* scale (12-ET) sacrifices a degree of harmonic purity by using notes with frequencies given by the geometric sequence $f_n = f_0 \cdot 2^{\frac{n}{12}}$, where f_0 is a reference frequency (e.g., A4 at 440 Hz). A single semitone function $s(x) = 2^{\frac{1}{12}} x$ expresses the constant frequency ratio between adjacent notes. Substitution for f_n in Equation (8) shows that the integer n is the semitone measure of the pitch relative to the reference note f_0. In 12-ET, a *tone* is just the composition $s \circ s$. This abolishment of the Pythagorean Comma means that enharmonically equivalent notes are now identical in pitch. However, their names are retained. Thus, for example, the 12 notes in an octave from C to 2C would be denoted

$$\mathrm{C} \overset{s}{\mapsto} \begin{Bmatrix} \mathrm{C}\sharp \\ \mathrm{D}^\flat \end{Bmatrix} \overset{s}{\mapsto} \mathrm{D} \overset{s}{\mapsto} \begin{Bmatrix} \mathrm{D}\sharp \\ \mathrm{E}^\flat \end{Bmatrix} \overset{s}{\mapsto} \mathrm{E} \overset{s}{\mapsto} \mathrm{F} \overset{s}{\mapsto} \begin{Bmatrix} \mathrm{F}\sharp \\ \mathrm{G}^\flat \end{Bmatrix} \overset{s}{\mapsto} \mathrm{G} \overset{s}{\mapsto} \begin{Bmatrix} \mathrm{G}\sharp \\ \mathrm{A}^\flat \end{Bmatrix} \overset{s}{\mapsto} \mathrm{A} \overset{s}{\mapsto} \begin{Bmatrix} \mathrm{A}\sharp \\ \mathrm{B}^\flat \end{Bmatrix} \overset{s}{\mapsto} \mathrm{B} \overset{s}{\mapsto} 2\mathrm{C}$$

By convention, the designation of a particular chromatic note as sharp or flat is now determined by the key. Although Equal Temperament scales solve the problem of key modulation, Duffin [7] and others argue that the abandonment of more natural tuning systems has had a deleterious effect on the richness and variety of musical harmonies. According to Jeans [6], "Some regard it as a defect of equal temperament tuning that it obliterates the different characteristic qualities of the various keys." He goes on to say that "... equal temperament tuning has resulted in much of the music of the earlier masters not being heard tonally as it was intended to be heard ...". At least, the choice of 12 notes yields a sequence which reasonably approximates the harmonically purer tones of Pythagorean and Just Intonation tunings. For example, in 12-ET the interval from C to G, $2^{\frac{7}{12}} \approx 1.498$, is extremely close to the Pythagorean ratio of $\frac{3}{2} = 1.5$.

53-Tone Equal Temperament Scales

By the use of continued fractions (see [8]), good rational approximations of irrational numbers can be found which lead to divisions of the octave which accurately approximate the harmoniously purer intervals determined by rationals. For example, it has been known since ancient times that 53 Pythagorean fifths is very close to 31 octaves, a fact which lead many to investigate 53-tone scales. Holder [9] noted that tempering a scale into 53 equal frequency ratios of $2^{1/53}$ yielded a scale (53-ET) which quite accurately approximates certain Just Intonation intervals (page 79).

Closing Remarks

It should be clear from the examples herein that the number of possible musical scales is very large, as is too the number of ways of constructing them. A quite extensive study of Greek, Arabian, Indian, Javanese, Chinese, and Japanese scales can be found in a paper of Alexander J. Ellis [10]. As Ellis notes, "Of course practice preceded theory in music". Ellis stresses the importance to music of what he calls "judgement of the ear."

Suggested Exercises for Students

1. In the Pythagorean scale, show that the interval from B to D$^\flat$ is $p^{-1} \circ \tau$. Calculate the measure of this interval in cents.

2. Verify that the compositions around both the inner and outer hexagons in Figure 7 commute to the identity map.

3. Show that the frequency relation $f_X \sim f_Y \iff f_X = 2^k f_Y$ for some integer k is an equivalence relation.

4. Show that the MIDI number of a note in the 12-ET scale is an integer.

5. Find the note with MIDI number 127.

6. Find a simple reason why, as a directed graph, Figure 10 can contain no Euler circuit (i.e., a circuit taking all edges exactly once).

7. Show that in a commutative diagram, commutativity of the cycles is sufficient for commutativity of the diagram.

8. Let $\tau(x) = 2^{\frac{1}{6}}x$ be a tone in the Equal Temperament scale. Find the function h such that the composition $h \circ \tau$ represents a transposition upwards by a fifth.

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