Guitar Chord Sequences Totally Ordered by Pitch

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Introduction

A permutation f of the set of 3-note voicings of the diatonic chords is found which preserves open/closed parity and rotates inversions. When lifted to $\mathbb{Z}$, the forward orbits of f are infinite ascending-pitch sequences of voicings. It follows that if just one diatonic open/closed chord is known, anywhere on say, a guitar fretboard, then totally ordered sequences of ascending and descending voicings of the same parity can be calculated, iteratively, without reference to dictionaries of chord fingerings. As an application, tables of open and closed chords for various major scales over different regions of the guitar fret board are generated.

Diatonic Triads

The set of notes of the C major diatonic scale is $S = \{0, 1, 2, 3, 4, 5, 6\}$, where C $= 0$, D $= 1$, E $= 2$, etc. A subset of S containing exactly three elements is called a *triad.* Let $\binom{S}{3}$ denote the set of triads. This notation derives from the fact that the number of triads in a set of 7 elements is determined by the binomial coefficient $\binom{7}{3} = 35$. Recall that any 3 element subset determines $3! = 6$ permutations, called *voicings* in music. The total number of voicings in the C Major scale is therefore $3!\binom{7}{3} = 210$.

Definition 1. A 3 note voicing is called *diatonic* if it is a permutation of one of the 7 *stacked third* forms:

$$\{(i, i+2, i+4) : i \in Z_7\} \tag{1}$$

I.e., a 3 note chord voicing is diatonic if its stacked third form is a concatenation $I_1 \oplus I_2$ of intervals $I_1 = [i, i+2]$ with $I_2 = [i+2, i+4]$ both of which are musical thirds.

Remark. It is easy to verify that a 3 note voicing has no more than one stacked third form In this form, the integers i, $i+2$ and $i+4$ (modulo 7) are called respectively, the *root* the *third* and *fifth* of the voicing . Since these integers are unique, the integer i serves to determine the letter *name* of the chord.

Definition 2. The *quality* of a 3 note chord voicing is

1. *major* if its stacked third form is $M_3 \oplus m_3$.
2. *minor* if its stacked third form is $m_3 \oplus M_3$.
3. *diminished* if its stacked third form is $m_3 \oplus m_3$.

Every musician knows that the 3 note stacked third forms of the C major scale are CEG, DFA, EGB, FAC, GBD, ACE, and BDF. Of these, the forms C, F and G are major, A,D and E minor and B diminished.

Example 1. Show that the voicing DBF is diatonic.

Solution

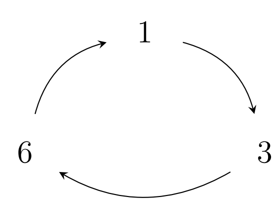
$$\begin{aligned}\{D, B, F\} &= \{1, 6, 3\} \\ &= \{6, 1, 3\} \\ &= \{6, (6+2) \bmod 7, (6+4) \bmod 7 \\ &= \{B, D, F\}\end{aligned}$$

Hence the voicing DBF is a permutation of the diatonic voicing BDF which has stacked third form with root B, third D and fifth F. Since the intervals BD and DF are both minor thirds, the chord B diminished and is written B° to indicate this fact.

Definition 3. The *parity* of a 3 note diatonic voicing is said to be

1. *even* (or in music *closed*) if it can be obtained from its stacked third form in an even number of transpositions.
2. *odd* (or in music *open*) otherwise.

Example 2. The open voicings of B°(= 613) are 613, 361 and 136. These happen to correspond to clockwise directed paths of length 2 in the directed graph



The *odd* voicings 631, 163, and 316 correspond to the reverse paths.

On a guitar with standard tuning, *closed* voicings, because of their narrower interval spacing, can be conveniently played with fingers on adjacent strings. Open voicings cannot.

Voicings are also classified according to their base note.

Definition 4. A voicing is said to be of:

1. *root position* if its base note is the root.

2. *first inversion* if its base note is the third.

3. *second inversion* if its base note is the fifth.

Example 3. The following table classifies the 3! voicings of the B diminished chords according to their parity and quality.

Table 1: B Diminished Triad Constructions

Perm.	Chord	Formula*	Intervals
613	closed root position	$1 - \flat 3 - \flat 5$	$m_3 \oplus m_3 = d_5$
136	closed first inversion	$\flat 3 - \flat 5 - 1$	$m_3 \oplus d_5 = M_6$
361	closed second inversion	$\flat 5 - 1 - \flat 3$	$d_5 \oplus m_3 = M_6$
631	open root position	$1 - \flat 5 - \flat 3$	$d_5 \oplus M_6 = m_{10}$
163	open first inversion	$\flat 3 - 1 - \flat 5$	$M_6 \oplus d_5 = m_{10}$
316	open second inversion	$\flat 5 - \flat 3 - 1$	$M_6 \oplus M_6 = d_{12}$

*The formula column shows the pitch of each note relative to the root. For example, in the permutation 361 = FBD the fifth F = 3 is a diminished fifth ($\flat 5$) or tritone above the root B and the third $D = 1$, a minor third above B and so the formula for this voicing is $\flat 5 - 1 - \flat 3$.

Permutations on $\mathcal{D}$

The set $\mathcal{D}$ of $42 = 3! \times 7$ of 3-note diatonic voicings splits evenly into a set $\mathcal{D}_{\mathcal{O}}$ of 21 open voicings and $\mathcal{D}_{\mathcal{C}}$ of 21 closed voicings. Both of these sets are further partitioned among the 3 qualities (root position, 1st inversion, 2nd inversion). It follows that $\mathcal{D}$ is a disjoint union of the following 6, nonempt sets

1. Root position open voicings: $O_1 = \mathcal{D}_{Or} = \{(i, i+4, i+2) : i \in \mathbb{Z}_7\}$,

2. First inversion open voicings : $O_2 = \mathcal{D}_{O1} = \{(i+2, i, i+4) : i \in \mathbb{Z}_7\}$.

3. Second inversion open voicings: $O_3 = \mathcal{D}_{O2} = \{(i+4, i+2, i) : i \in \mathbb{Z}_7\}$.

4. Root position closed voicings: $C_1 = \mathcal{D}_{Cr} = \{(i, i+2, i+4) : i \in \mathbb{Z}_7\}$,

5. First inversion closed voicings : $C_2 = \mathcal{D}_{C1} = \{(i+2, i+4, i) : i \in \mathbb{Z}_7\}$.

6. Second inversion closed voicings: $C_3 = \mathcal{D}_{C2} = \{(i+4, i, i+2) : i \in \mathbb{Z}_7\}$.

Theorem 1. There exists a permutation $f : \mathcal{D} \to \mathcal{D}$ such that

1. $f|_{O_i}$ is a bijection onto $f|_{O_{i+1}}$, $\quad 1 \leq i \leq 3$

2. $f|_{C_i}$ is a bijection onto $f|_{C_{i+1}}, \quad 1 \leq i \leq 3$

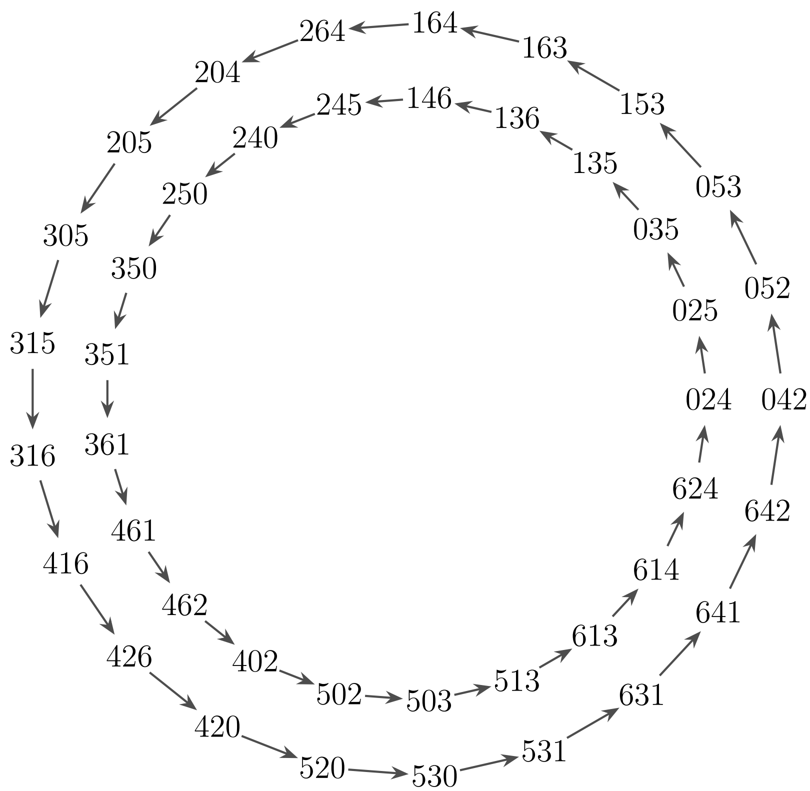
I.e., f induces, in a natural manner, a permutation on the partition of $\mathcal{D}$ which is both parity preserving and quality rotating.

Proof. Let $P(S,3)$ be the set of all voicings in S. Let $f : \mathcal{D} \to P(S,3)$ be the map which raises the fifth of a diatonic voicing by one scale division. We will show that the restriction $f|_{C_1}$ is a parity preserving bijection onto $f|_{C_2}$. The proof for the other 5 sets in the partition is similar and is left to the reader. Suppose that $x \in C_1$. Then for some $i \in \mathbb{Z}_7$, $x = (i, i+2, i+4)$ and $f(x) = (i, i+2, i+5)$. If $j = i+5$ then $f(x) = (j+2, j+4, j) \in C_2$. Thus, $f|_{C_1}$ has image in C_2. Since $j = i+5 \pmod 7$, the stacked third form of $f(x)$, namely $(j, j+2, j+4)$ is just a rotation of x by an interval of a sixth – a parity preserving bijection. Now let $g : \mathcal{D} \to P(S,3)$ be the map which lowers the the root of a diatonic voicing by one scale division. We claim that $g|_{C_1}$ is the inverse of $f|_{C_1}$. Let $y \in C_2$. Then for some $j \in \mathbb{Z}_7$ $y = (j+2, j+4, j)$ and so $g(y) = (j+2, j+4, j-1)$. Let $i = j+2$ then $g(y) = (i, i+2, i-3) = (i, i+2, i+4) \in C_1$. Since $i = j+2 \pmod 7$, $g(y)$ is a rotation of y by an interval of two scale degrees (a musical third) and so $f|_{C_2}$ is a parity preserving bijection onto C_1 and $g|_{C_2} = f|_{C_1}^{-1}$. □

Total Orders on $\mathcal{D}_{\mathcal{O}}$ and $\mathcal{D}_{\mathcal{C}}$

Since f is permutation of $\mathcal{D}$ it function digraph consists of disjoint cycles. Since no open voicing can map to a closed voicing, there must be at least two cycles. Computation shows that there are exactly 2 (See Figure 1).

Figure 1: Drawing of the Function Digraph of f



Definition 5. Let $x \in \mathcal{D}$. The set $\{x, f(x), f^2(x), \ldots\}$ is called the *forward orbit* of x

Because the addition in the definition of f is modulo 7, position in the orbit of a diatonic voicing x does not determine an order on $\mathcal{D}$. Indeed, no order can contain a cycle. Only if a set A is infinite can the digraph of a function on A not contain a cycle or loop. In order to define a total orders on $\mathcal{D}_\mathcal{O}$ and $\mathcal{D}_\mathcal{C}$ we enlarge these sets to be infinite. All that is needed to achieve these is to take higher octaves into account by replacing addition $\pmod 7$ by ordinary addition in the definition of f. For any note i, we let $i', i'', \ldots$ denote the notes one octave, 2 octaves etc., above i. With this convention the forward orbits of $x \in \mathcal{D}$ extend indefinitely. Moreover, since each iteration of f, represents an increase in pitch, an order can be defined by $x \lesssim y \iff y = f(x)$. The transitive closure of this order

$$x \lesssim f^k(x), \quad k > 0$$

defines a total order by pitch on the infinite forward orbit of x. This order preserves the parity of x

Application to Guitar Voicings.

To illustrate these ideas a table of E minor open voicings will be generated. For compatibility with other sources we will henceforth use a 1-based numbering {1,2,3,4,5,6,7 } for the C major scale. Chord diagrams will use standard tuning (E_2, A_2, D_3, G_3, B_3, E_4), and, following common practice, guitar music will be notated one octave higher than the true pitch. Using 1-based numbering, the lowest open E minor chord voicing on a guitar (E2B2G3) will now be denoted 375 The forward orbit of $x = 375$ under f determines the following order:

$$375 \lesssim 31'5 \lesssim 31'6 \lesssim 41'6 \lesssim 42'6 \lesssim 42'7 \lesssim 52'7 \lesssim 53'7 \lesssim 53'1' \lesssim 63'1' \lesssim 64'1' \lesssim 64'2' \lesssim 74'2' \lesssim 75'2' \lesssim 75'3' \lesssim 1'5'3' \lesssim 1'6'3' \lesssim 1'6'4' \lesssim 2'6'4' \lesssim 2'7'4' \lesssim 2'7'5' \lesssim 3'7'5' \lesssim 3'1''5' \lesssim$$

Recall that the map f raises the fifth note of a diatonic triad by one scale degree. In the case of a major scale (see [5]), this means by 1 semitone in the case of the iii and vi chords and by 2 semitones otherwise. Similarly, the inverse operation's lowering of the root by one scale degree means by 1 semitones in the case of chords I or IV, and by 2 semitones otherwise. By iterating these simple operations on a given musical instrument, any open, or closed diatonic triad voicing, can be used to sequences of a fixed parity, totally ordered by pitch.

Figure (3) depicts an pitch ordered sequence of 24 voicings from the scale of C major. The particular choice of fingering allows all of the chords to be played comfortably on the first three frets of a guitar. The sequence can easily be extended to arbitrary higher frets and, As shown in [6], many other pitch equivalent fingerings are possible. A pitch ordered sequence of 48 voicings for the key of G major can be found in Appendix 1.

1. As noted in the proof of theorem 1, the chords cycle (modulo 7) through the scale in sixth's (equivalently backwards in thirds) : $1, 6, 4, 2, 7, 5, 3, 1, 6, 4, \ldots$

2. Each column contains a harmonization of C major by chords in a fixed inversion.

3. Distinct columns have distinct inversions.

Figure 2: C Major Open Voicings - Sheet

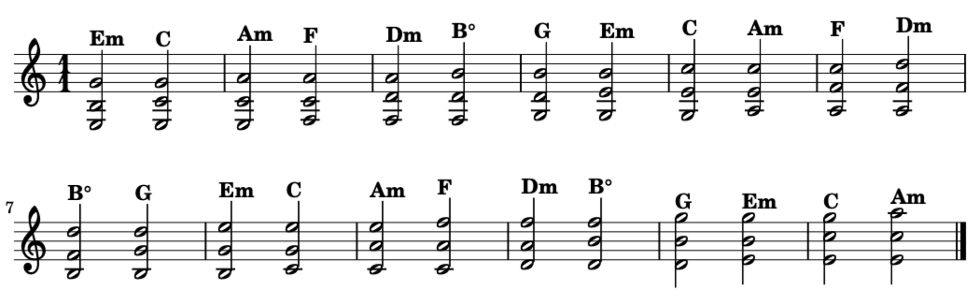
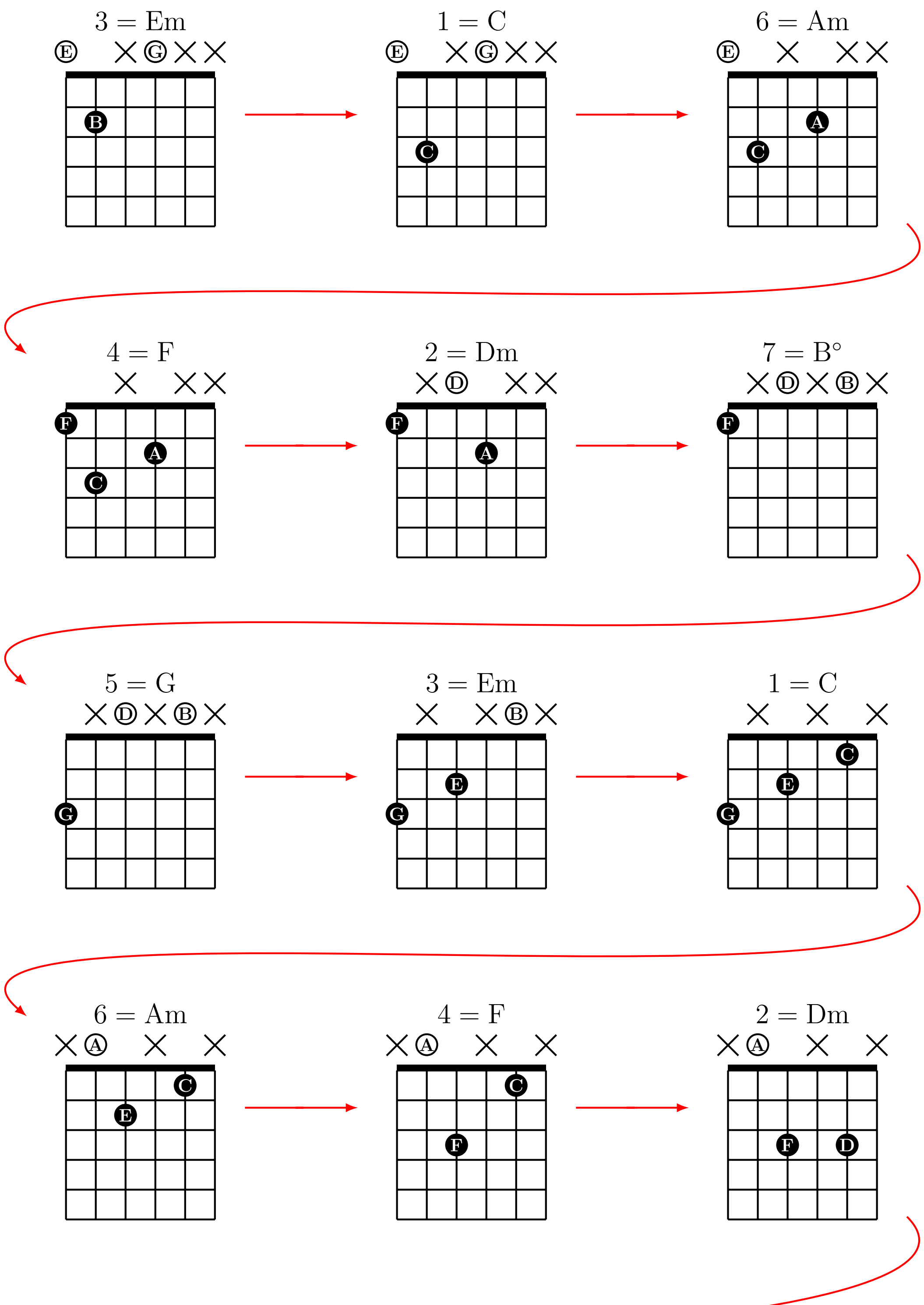


Figure 3: C Major Open Triads on Frets 0 to 4
Totally Ordered by Increasing Pitch



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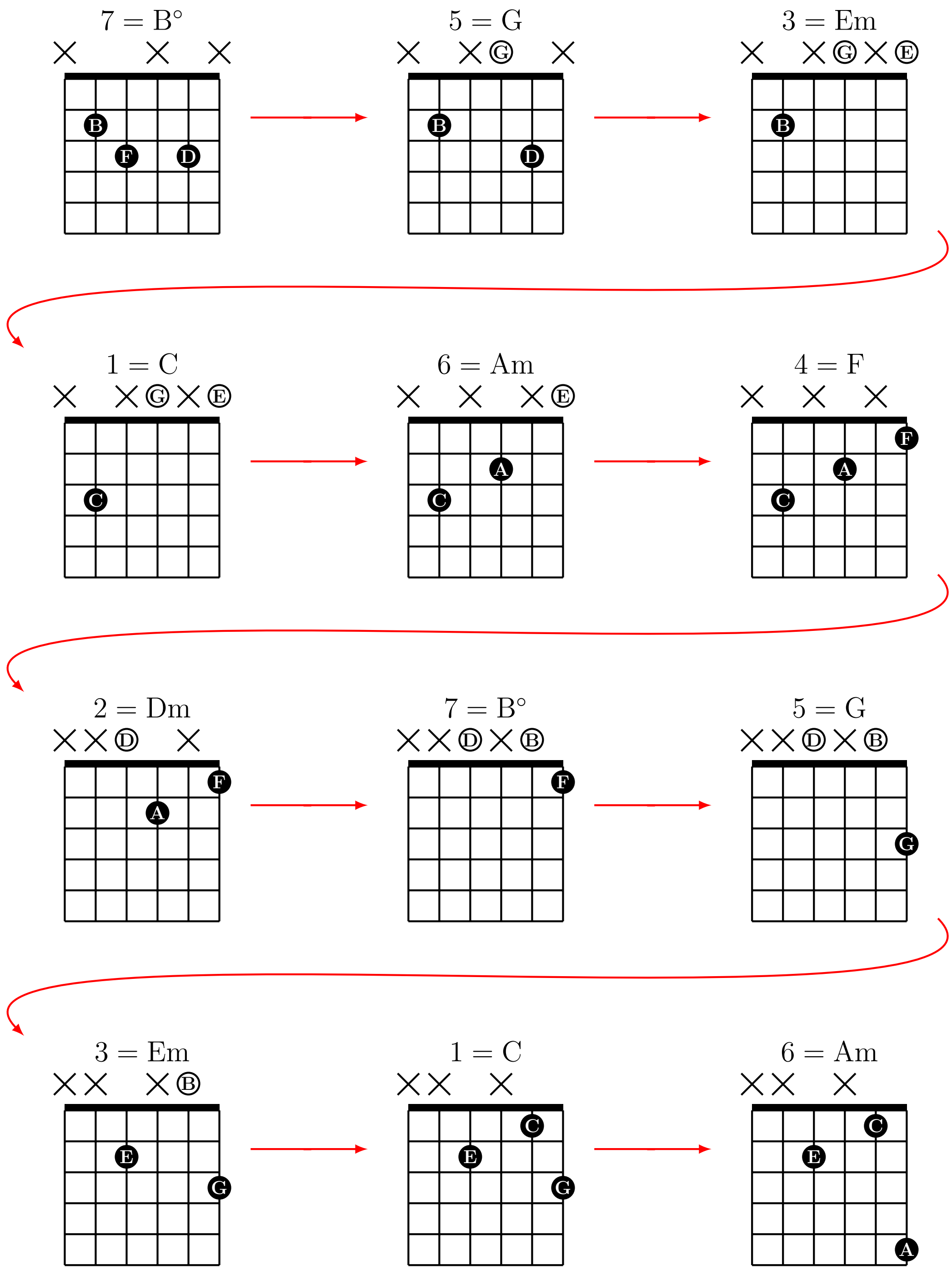
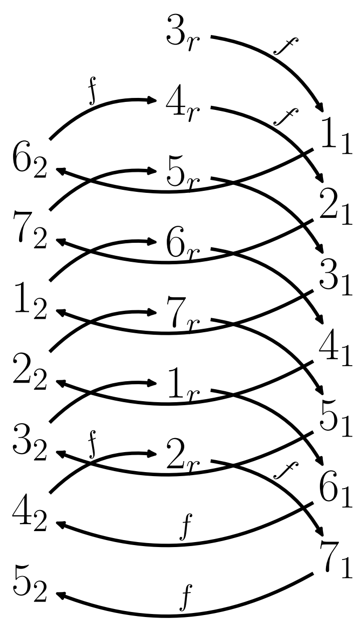


Other vizualizations

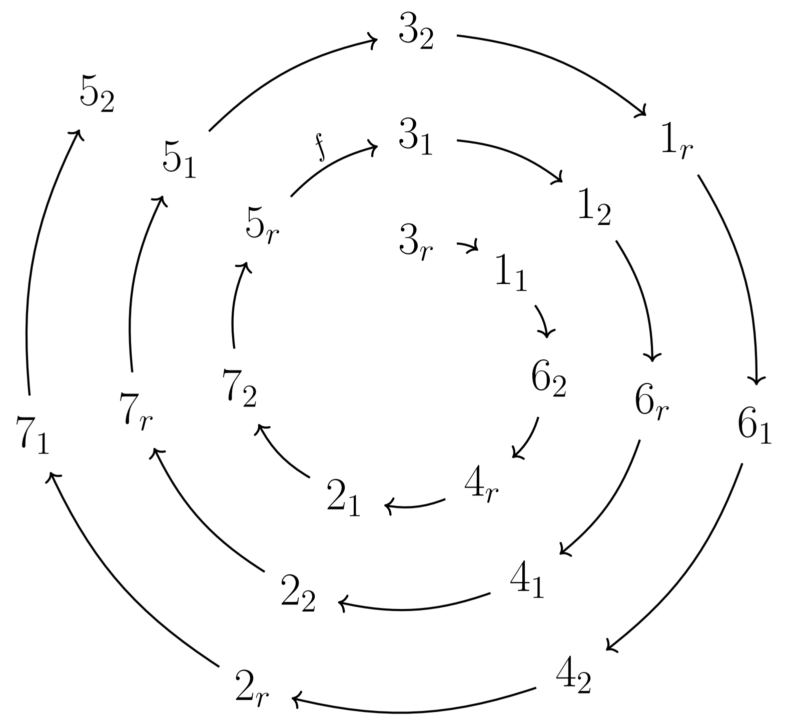
Chord sequences generated by the order $\lesssim$ can be drawn (as suggested by the table) as helices See Figure 4, with each turn containing the 3 qualities (root, first inversion, second inversion) in the same order.

Figure 4: Orbit of Open Em – Root Voicing
Drawn as a Cylindrical Helix



Alternatively (see Figure 5) in two dimensions, the chords of the ordered sequence can be placed along a spiral with 7 chords per rotation, ordered in steps of 2 notes as shown. Root positions and inversions are indicated by the subscripts.

Figure 5: Orbit of Em Root Voicing
Drawn an Archimedian Spiral



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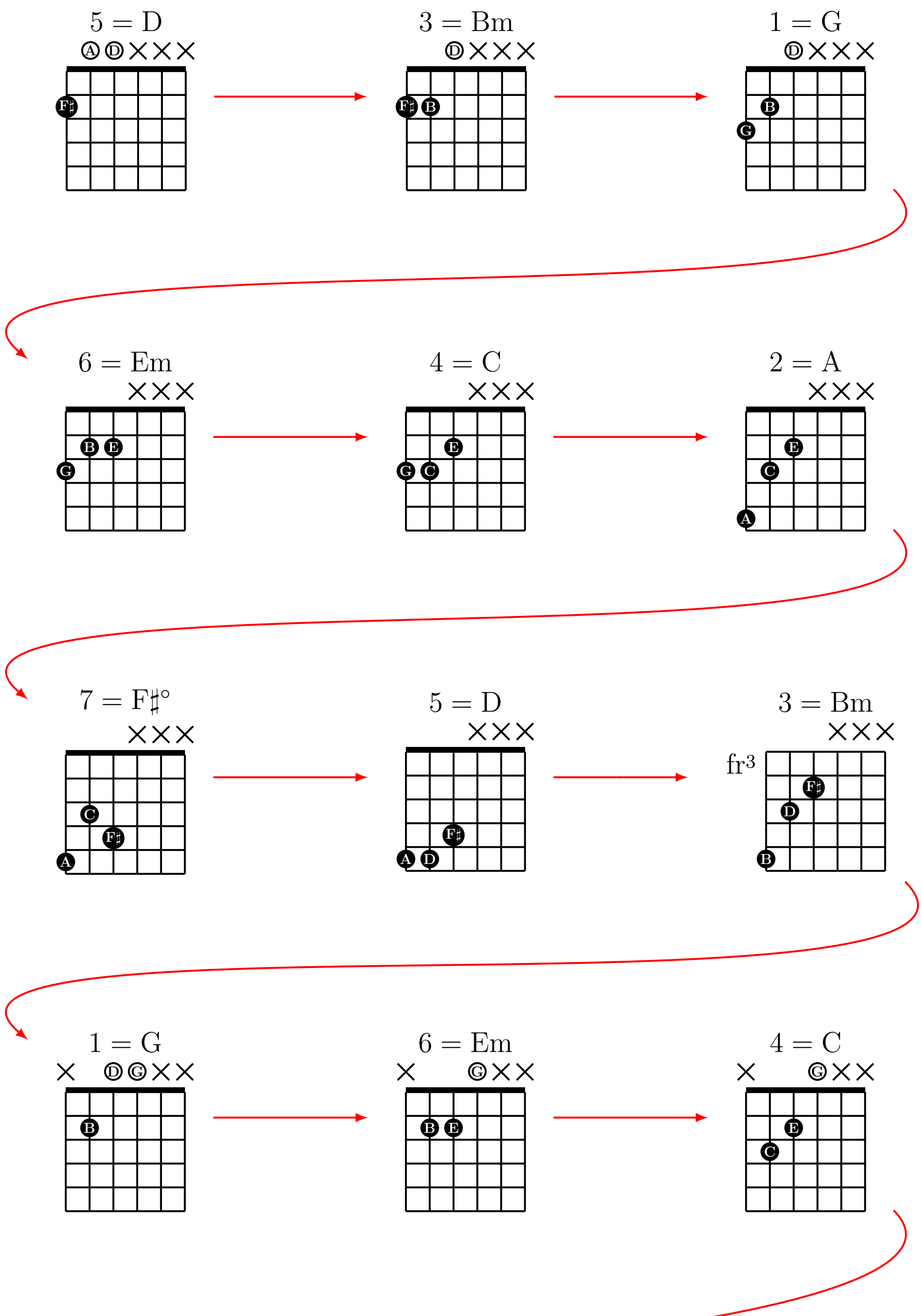
Appendix 1. G Major Closed Voicings Ordered by Pitch Ascending



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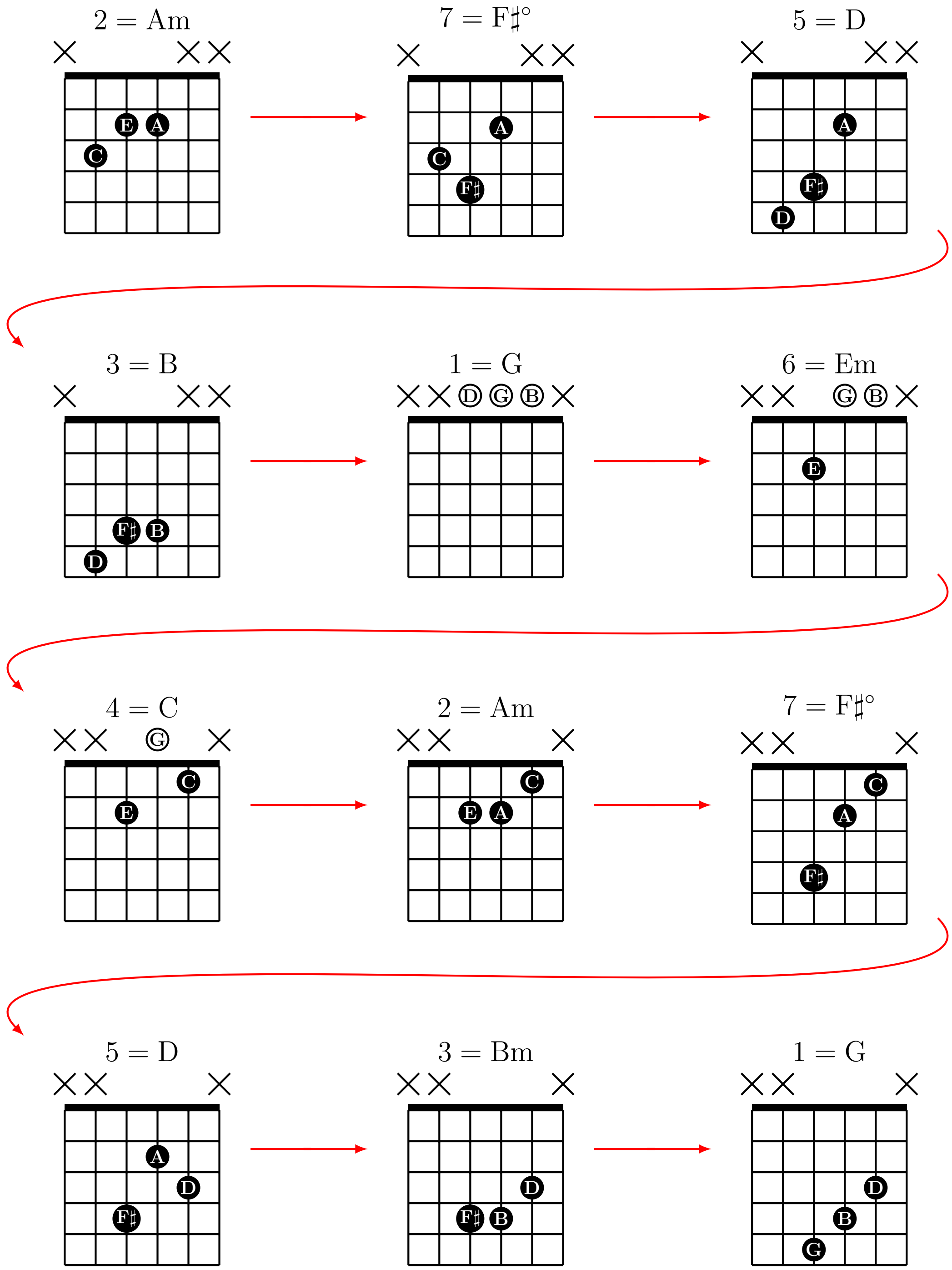

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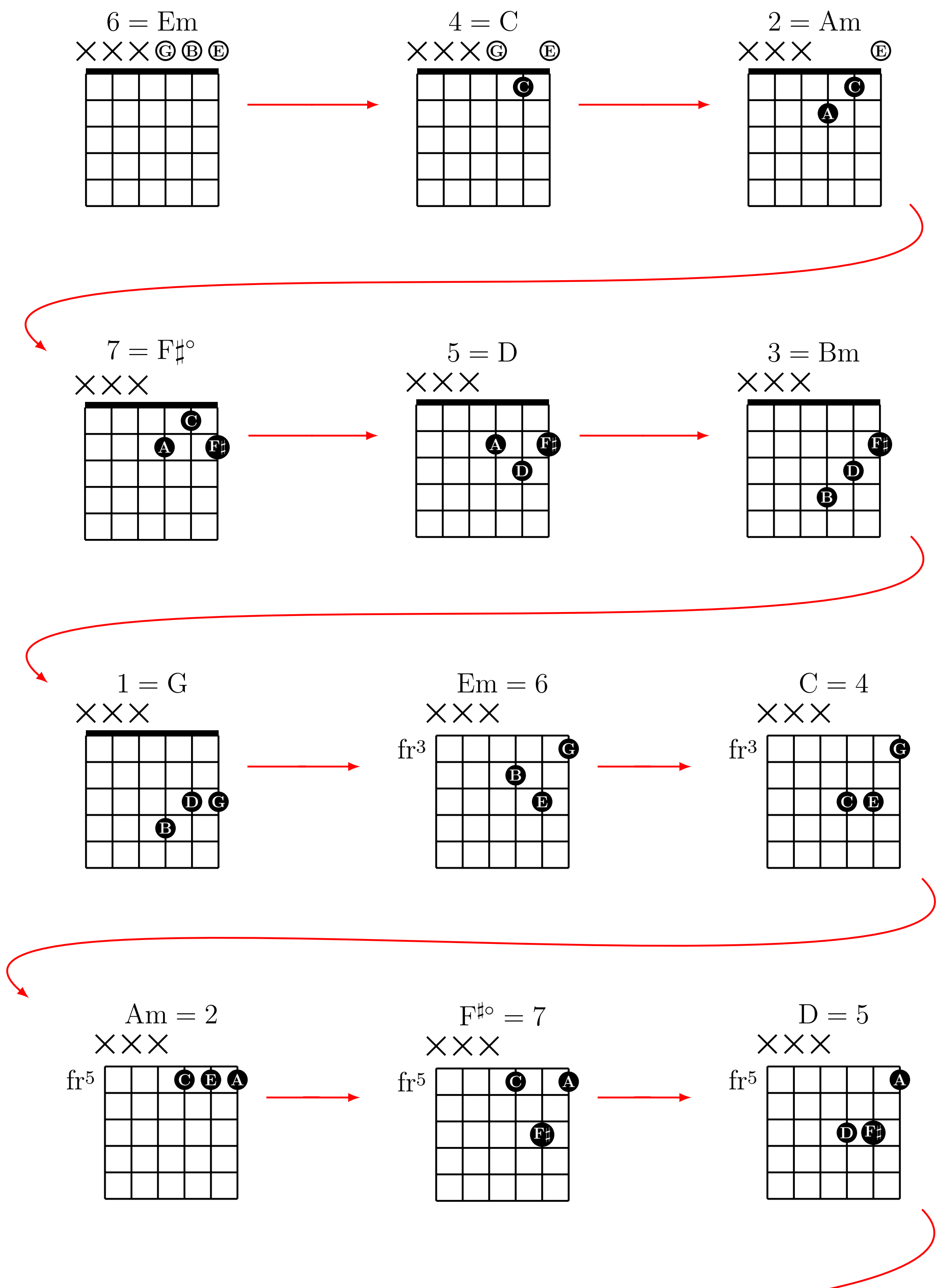

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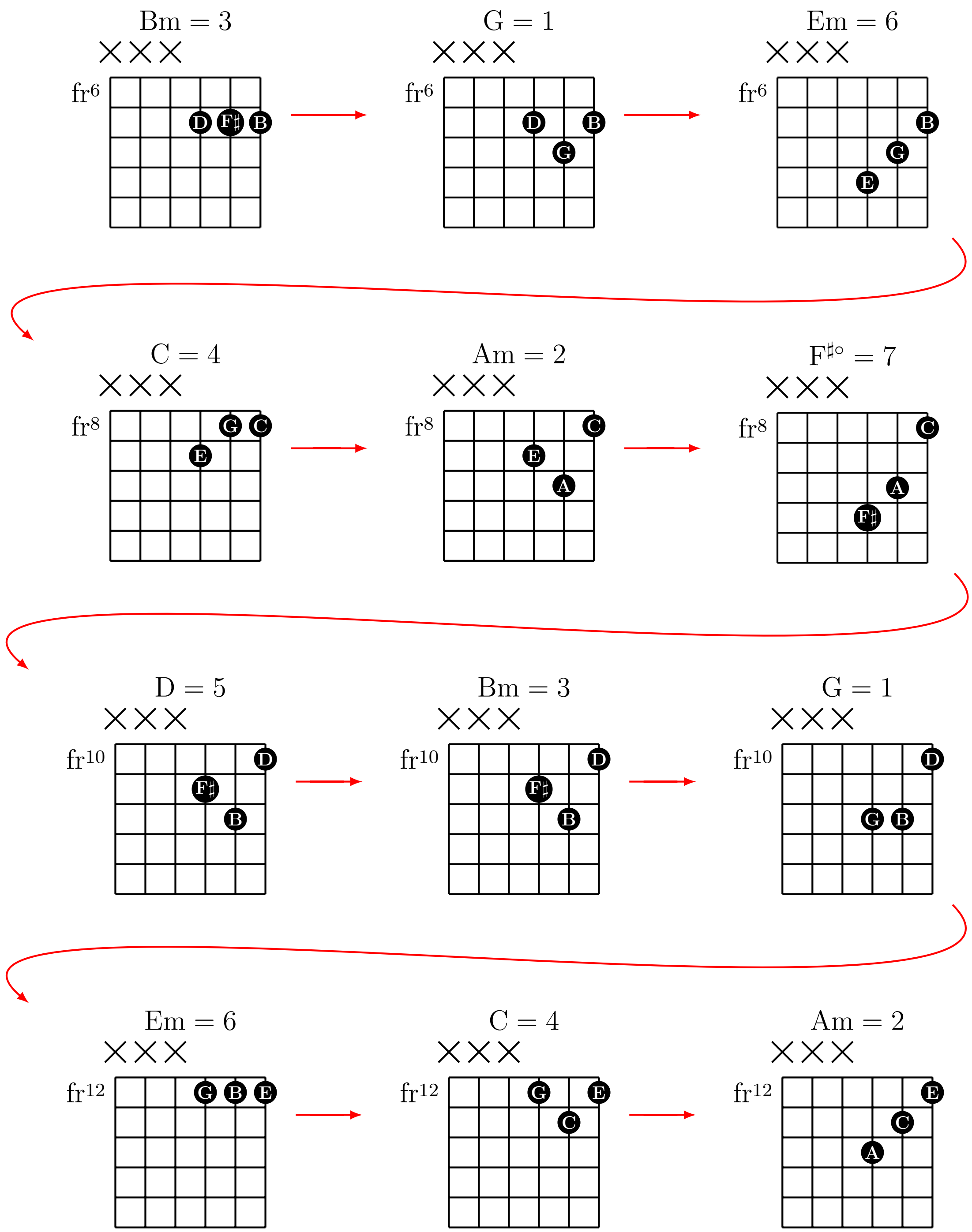


Appendix 2. Pitch Equivalent Guitar Fingerings of the B Diminished Open Triads

Remarks

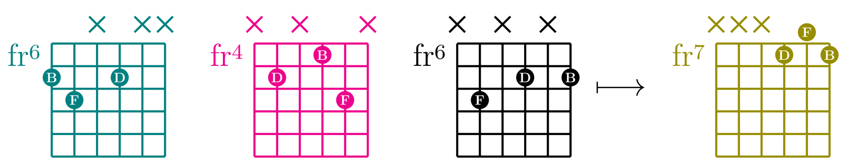
The following is a 5×4 table of some commonly used voicings of the B° open triads on a 21 fret guitar. Let $X(i, j)$ be the chord in row i and column j, so that, for example, the top left and bottom right chords of the table will be denoted $X(1, 1)$ and $X(5, 4)$ respectively. Notice that:

1. Chords on a given row fall in roughly the same range of frets.

2. Each row contains a root, first inversion and second inversion triad of the open B°. The fourth chord is an upward pitch translation by one octave of the first chord. Thus, for example, $X(3, 4)$ is one octave higher than $X(3, 1)$ and of the same inversion.

3. Pitch order is constant on fingerings of the same color (located on the diagonals). Since there are eight colors in the table, there are exactly eight pitch classes. It is natural to refer to fingerings of the same color as being *pitch equivalent.* Clearly all fingerings in a given pitch class will have the same inversion.

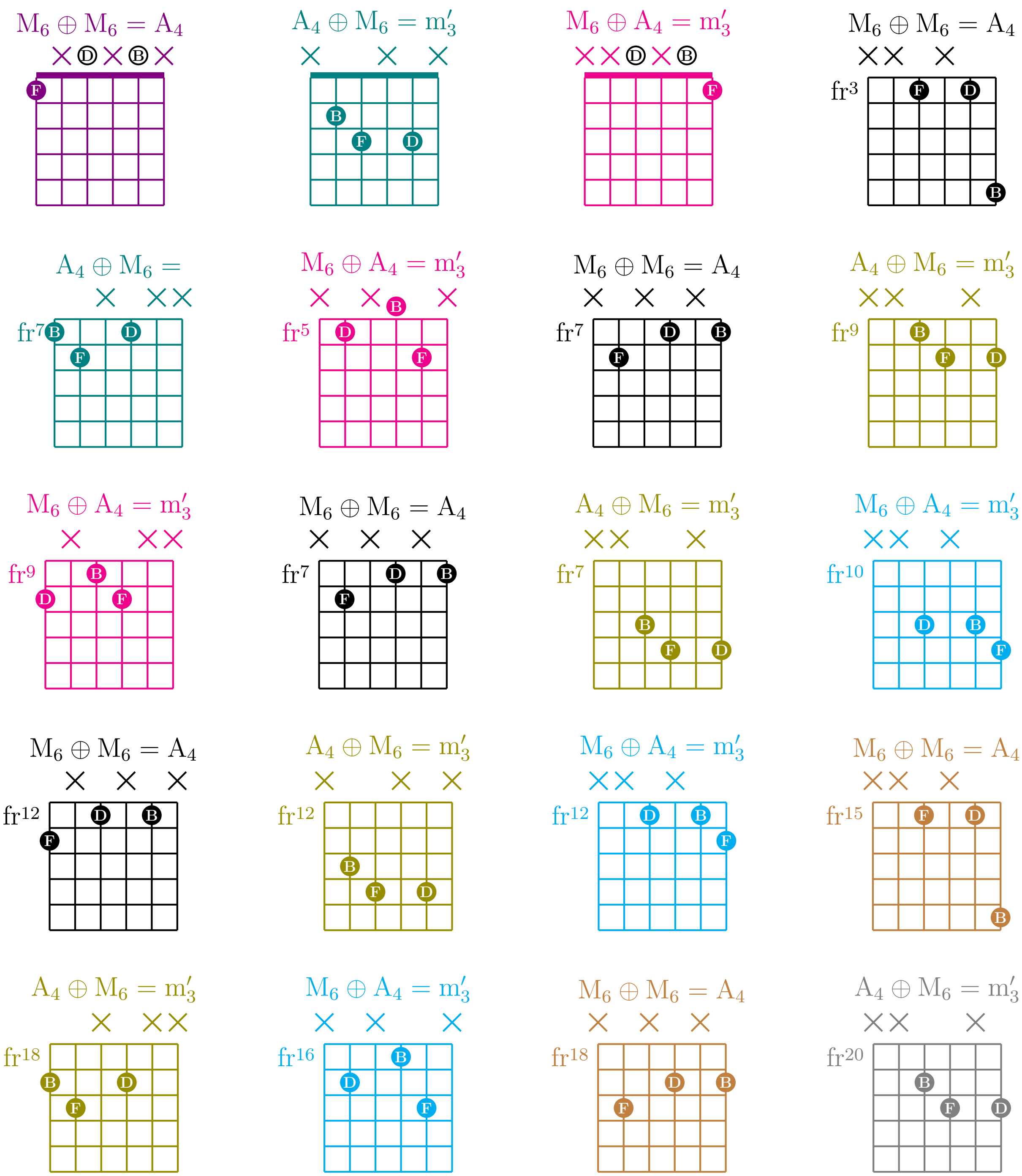
4. The highest notes in each of the first three chords of a given row, together, form an inversion of one of the three B diminished closed triads located on the first three strings. For example, here are the first three diminished open chords in row two, followed by the corresponding B diminished closed inversion on the first three strings:



   Different rows of the table generate different B diminished closed triads in this manner. Conversely, each B diminished closed triad on the first three strings generates a unique triple of open B diminished open triads whose highest notes comprise the closed triad. Thus there is a one-to-one correspondence between B diminished closed triads on the first three strings and the triples of B diminished open triads. Knowledge of this correspondence may facilitate memorization of the open triads.

5. Since there are only three odd permutations of three notes, the chords in row $i + 3$ will be an octave higher than in row i and of the same inversion.

6. The fingerings of diminished chords in other keys can now be obtained by a pitch translation of the entire table.



Some Suggestions for Practice

The chords can be practiced

1. One row at a time (so as to remain within a given fret range). Before and after each row, the closed chord determined by the highest notes of the three open chords can also be played.

2. One column at a time.

3. All chords of a given color.

4. In order of pitch increasing and pitch decreasing. This can be done by playing along all paths of length 8 connecting $X(5,4)$ to $X(1,1)$. Notice that these paths contain every color exactly once.

5. Interleaved with musically related chords.