A Mathematical Introduction to Melody

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1. Introduction

Algebraic music theory rests on the observations that the set $\mathcal{I}$ of intervals forms a $\mathbb{Z}$-module (or free abelian group) under concatenation and that the set of pitches forms an affine space (or torsor) over this module. A change of key is simply a shift of origin.

Within this framework a melody is a walk in the affine space subject to well-defined local constraints (conjunct motion, leap compensation, tendency-tone resolution, diatonic membership, contour, range, etc.). A chord is expressed, via Chasles' identity, as a vector sum of intervals; the same language yields a clean description of closed and open voicings, pitch-equivalent fingerings, and the affine subspaces that correspond to whole-tone scales, augmented triads, diminished-seventh chords and tritones.

The present notes develop these ideas in elementary terms, supply the necessary multiplication and inversion tables, give explicit fretboard realisations of the principal intervals and dyads, and illustrate the theory with a detailed analysis of the traditional melody and harmony of *Greensleeves.* An appendix collects interval-stack constructions for the most common triads, seventh chords and ninth chords.

Some knowledge of modern algebra is assumed.

Contents

2. Group Structure

In the usual 12-tone equal-temperament scale [13] the set of intervals is

$$\mathcal{I} = \{m_2, M_2, m_3, M_3, P_4, A_4, P_5, m_6, M_6, m_7, M_7, O\}.$$

These correspond to semitone counts modulo 12 of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Addition modulo 12 induces a concatenation operation $\oplus$ on $\mathcal{I}$ (See Table 1).

The following is well known to music theorists:

Theorem 1. *$(\mathcal{I}, \oplus)$ is an abelian group isomorphic to the cyclic group $\mathbb{Z}/12\mathbb{Z}$*

The proof can be found in any modern algebra text. For relevant definitions see references [17] and the elementary theory of cyclic groups [14]. Inverse elements and the order of each are listed in Tables 2 and Table 3

Table 1: Interval Concatenations

$\oplus$	m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7	O
m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7	O	m_2
M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7	O	m_2	M_2
m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7	O	m_2	M_2	m_3
M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7	O	m_2	M_2	m_3	M_3
P_4	A_4	P_5	m_6	M_6	m_7	M_7	O	m_2	M_2	m_3	M_3	P_4
A_4	P_5	m_6	M_6	m_7	M_7	O	m_2	M_2	m_3	M_3	P_4	A_4
P_5	m_6	M_6	m_7	M_7	O	m_2	M_2	m_3	M_3	P_4	A_4	P_5
m_6	M_6	m_7	M_7	O	m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6
M_6	m_7	M_7	O	m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6
m_7	M_7	O	m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7
M_7	O	m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7
O	m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7	O

Table 2: Group Inverses

Interval	m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7	O
Inverse	M_7	m_7	M_6	m_6	P_5	A_4	P_4	M_3	m_3	M_2	m_2	O

Notice that under inversion: $M \leftrightarrow m$, $P \leftrightarrow P$ and $A \leftrightarrow A$ and the interval numbers add to 9.

Since $\mathbb{Z}/12\mathbb{Z}$ is cyclic of order 12, the generators are precisely the elements of order 12, namely m_2, P_4, P_5 and M_7. Thus, for example, starting with any pitch, and adding P_5 repeatedly generates all pitches. The graph generated by P_5 and M_2 is shown in Figure 1.) The subgraph generated by P_5 with bold edges is known graph isomorphic to the circle of fifths.

3. $\mathbb{Z}$-module Structure

Definition 1. *A $\mathbb{Z}$-module $\mathcal{I}$ consists of:*

1. *An abelian group* $(\mathcal{I}, \oplus)$

Table 3: Orders of Group Elements

	m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7	O
Semitones k	1	2	3	4	5	6	7	8	9	10	11	0
gcd(k,12)	1	2	3	4	1	6	1	4	3	2	1	12
Order	12	6	4	3	12	2	12	3	4	6	12	1

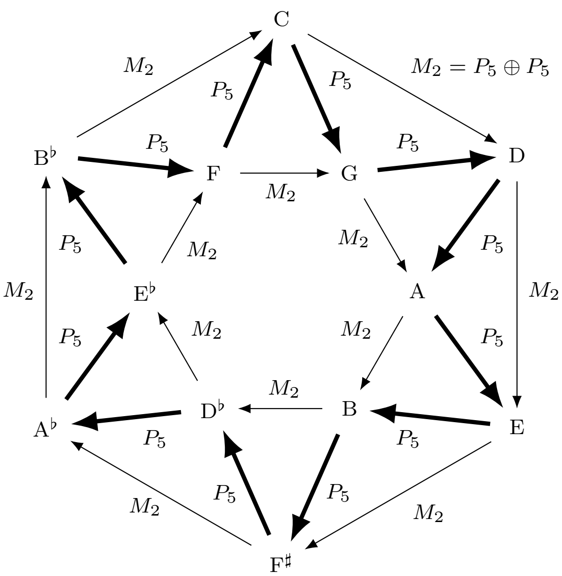


Figure 1: Graph Generated by P_5 and M_2

2. *A scalar multiplication map* $\mathbb{Z} \times \mathcal{I} \to \mathcal{I}$*, denoted by juxtaposition,* $(n, I) \mapsto nI$ *such that*

   (a) $n(I \oplus I') = nI \oplus nI'$.

   (b) $(n + m)I = nI \oplus mI$.

   (c) $n(mI) = (nm)I$

   (d) $1 \cdot I = I$.

A $\mathbb{Z}$-module generalizes the notion of a vector space, but with scalars drawn from the ring of integers $\mathbb{Z}$ rather than from a field. There is a one-to-one correspondence between $\mathbb{Z}$-modules and abelian groups. Specifically, an abelian group becomes a $\mathbb{Z}$-module by defining scalar multiplication via repeated addition. Conversely, if scalar multiplication is forgotten a Z-module possesses an abelian-group structure under $\oplus$.

Multiplication of an interval I by an integer n is defined by concatenation of n copies of I if $n > 0$ and of its inverse if $n < 0$. Naturally for any interval I, $0I = O$.

With this additional operation $\mathbb{Z} \times \mathcal{I} \to \mathcal{I}$ makes the abelian group of intervals $\mathcal{I}$ a $Z-$module. Some examples of scalar multiples of intervals are shown in table (4).

Table 4: Scalar Multiples

I	m_2	M_2	m_3	M_3	P_4	A_4	P_5	m_6	M_6	m_7	M_7	O
$2I$	M_2	M_3	A_4	m_6	m_7	O	M_2	M_3	A_4	m_6	m_7	O
$3I$	m_3	A_4	M_6	O	m_3	A_4	M_6	O	m_3	A_4	M_6	O
$4I$	M_3	m_6	O	M_3	m_6	O	M_3	m_6	O	M_3	m_6	O
$5I$	P_4	m_7	m_3	m_6	m_2	A_4	M_7	M_3	M_6	M_2	P_5	O
$6I$	A_4	O	A_4	O	A_4	O	A_4	O	A_4	O	A_4	O

If modulo 12 addition of semitones is replaced by integer addition, the set of intervals under concatenation becomes a free $\mathbb{Z}$-module, i.e., isomorphic to the infinite abelian group Z rather than to $\mathbb{Z}/12\mathbb{Z}$. In this infinite module, intervals of more than 12 semitones are called *compound*. Intervals not exceeding an octave are called *simple*. Examples are given in Table (5). Both viewpoints turn out to be useful in algebraic music theory.

Table 5: Compound Intervals

Compound Interval	Name	Semitones	Simple Equivalent
m_9	Minor 9th	13	$m_2 + O$
M_9	Major 9th	14	$M_2 + O$
M_{10}	Major 10th	16	$M_3 + O$
m_{10}	Minor 10th	15	$m_3 + O$
P_{11}	Perfect 11th	17	$P_4 + O$

Compound intervals cannot be inverted in the previous sense that an interval concatenated with its inversion must equal exactly one octave. Instead, inverses of the corresponding simple form are used.

4. Affine Structure

In physics, particle trajectories are modeled by sets of points in a three or perhaps four dimensional space with special properties. Forces and displacements are viewed as vectors "acting upon" points ([5]). No point is regarded as origin – the difference between geocentrism and helio-centrism, as verified by general relativity, is merely a philosophical choice ([12]).

Analogously, in music, melodies are modeled by sequences of notes (pitches) governed by certain constraints. In equal temperament scales, intervals are viewed as $\mathbb{Z}-$module elements "acting upon" notes. The set of notes has no canonical origin. By a change of key, any note can serve as tonic. The choice of tonic for a melody is therefore largely aesthetic.

Both situations are captured mathematically by the notion of affine space.

Definition 2. *Let $\mathcal{V}$ be a vector space with addition operation $\oplus : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. Let O denote the zero vector in V. An affine space A over $\mathcal{V}$ is a triple $(A, \mathcal{V}, +)$ consisting of a nonempty set A (of elements called points), and a function $+ : A \times \mathcal{V} \to A$, satisfying:*

1. *$a + O = a$, for every $a \in A$.*
2. *$(a + v) + v' = a + (v \oplus v')$, for every $a \in A$, and every $v, v' \in \mathcal{V}$.*
3. *For any two points $a, a'; \in A$, there is a unique vector $v \in \mathcal{V}$ such that $a+v = a'$.*

The first two properties are the usual defining properties of a (right) group action [16]. The elements of $\mathcal{V}$ are called *translations*, or *free vectors.* A group action on a set A is called *transitive* if for any two points $a, a' \in A$, there is a vector $v \in \mathcal{V}$ such that $a + v = a'$ and *free* if this vector is unique. Thus property (3) of an affine space specifies that the action is both transitive and free. An affine space over a module M is called an *$M-$torsor* or a *principal homogeneous space.* We work throughout with modules rather than vector spaces; the definitions carry over verbatim

In the musical case, the affine space is just a set of pitches S belonging to an equal temperament scale. The role of the vector space is played by the $\mathbb{Z}-$module of intervals (typically either $\mathbb{Z}$ or $\mathbb{Z}/12\mathbb{Z}$ under concatenation (stacking of intervals). This operation on intervals satisfies all the vector-space addition axioms. The existence of a unique interval connecting any two pitches. means that this action transitive and free. In summary, the set S of pitches is an M-torsor

4.1. Applications to Chords

Simple properties of chords can be described in terms of the affine space structure of their pitches. Consider, for example, the stacked third form CEG (see [18] for definition) of the C major chord. Adopting the convention that in an affine space, the unique free vector v such that $a+v = a'$ is denoted aa', we have that $E = C+CE$, $G = E + EG$ and $C = G + GC$. Hence,

$$C = E + EC = (G + GE) + EC = G + (GE \oplus EC).$$

By uniqueness of CG it follows that $CG = CE \oplus EG$. This totally trivial observation, is a special case of Chasle's identity which is true in any affine space. That any chord can be expressed as an interval stack is thus a simple expression of this more general result. See Figure (2) for vector diagrams of some common guitar chord voicings.

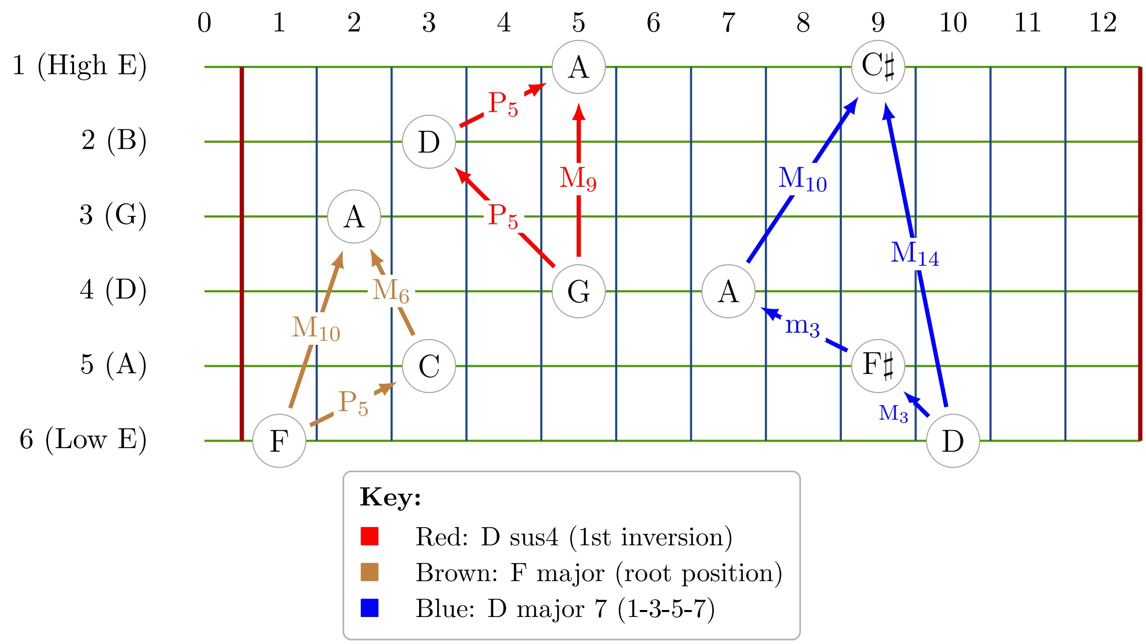


Figure 2: Vector Diagrams of some common chords

Table 6: Common Major Triad Constructions

Chord	Example	Construction
closed root position	1-3-5	$M_3 \oplus m_3 = P_5$
closed first inversion	3-5-1	$m_3 \oplus P_4 = m_6$
closed second inversion	5-1-3	$P_4 \oplus M_3 = M_6$
open root position	1-5-3	$P_5 \oplus M_6 = M_{10}$
open first inversion	3-1-5	$m_6 \oplus P_5 = m_{10}$
open second inversion	5-3-1	$M_6 \oplus m_6 = P_{11}$

Table 7: Minor Triad Constructions

Chord	Example	Construction
closed root position	1-b3-5	$m_3 \oplus M_3 = P_5$
closed first inversion	b3-5-1	$M_3 \oplus P_4 = M_6$
closed second inversion	5-1-b3	$P_4 \oplus m_3 = m_6$
open root position	1-5-b3	$P_5 \oplus m_6 = m_{10}$
open first inversion	b3-1-5	$M_6 \oplus P_5 = M_{10}$
open second inversion	5-b3-1	$m_6 \oplus M_6 = P_{11}$

Table 8: Diminished Triad Constructions

Chord	Example	Construction
closed root position	1-b3-b5	$m_3 \oplus m_3 = d_5$
closed first inversion	b3-b5-1	$m_3 \oplus d_5 = M_6$
closed second inversion	b5-1-b3	$d_5 \oplus m_3 = M_6$
open root position	1-b5-b3	$d_5 \oplus M_6 = m_{10}$
open first inversion	b3-1-b5	$M_6 \oplus d_5 = m_{10}$
open second inversion	b5-b3-1	$M_6 \oplus M_6 = d_{12}$

Tables (6), (7) and (8) show interval stacks for the major, minor and diminished triads. Guitar voicings for these can be found in [19].

For the construction of chord fingerings, knowledge of all intervals from a fixed base point on the guitar fretboard to any other point is essential. Figure 3.) shows all intervals (modulo 12) from base note A = O. The reader should transpose this to other basepoints as needed[1].

[1]The standard guitar is tuned in successive perfect fourths except for the interval between the third (G) and second (B) strings, which is a major third. Consequently, when an interval vector crosses this exceptional string its orientation may change. Vector addition of successive intervals will also fail to obey the parallelogram law of a flat (Euclidean) lattice. This local breakdown of Euclidean additivity is analogous to the curvature of space-time induced by a gravitational field.

At any fixed fret the collection of possible interval vectors emanating from a given note forms a discrete "star." That star plays a rôle similar to the tangent space of a differentiable manifold: it records all infinitesimal (here, one-fret or one-string) displacements available at that point, while the anomalous tuning of the third string introduces a non-vanishing "curvature" that prevents these local vectors from assembling into a globally flat geometry.

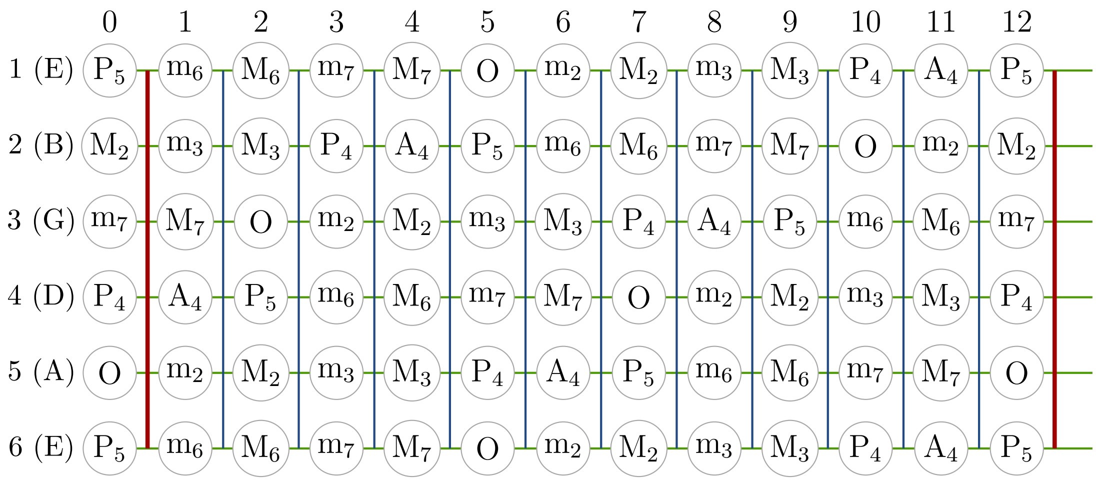


Figure 3: Fretboard Intervals (modulo 12) Relative to Base Note A = O

Using Figure (3) it is easy to construct vectors of any interval starting on any string. For example, Figure (4) shows some major sixth dyads[2] starting fret 5.

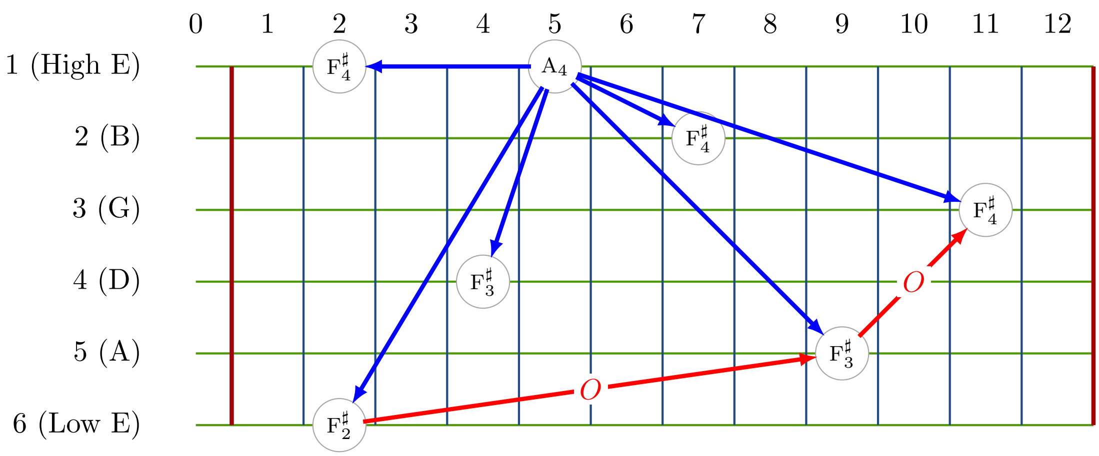


Figure 4: Major sixth vectors on the guitar fretboard
■ Blue = M_6 vector; ■ Red = Octave vector

[2]Mathematically (affine / directed) the descending arrows are minor thirds or equivalently negative major sixths. According to the prevailing guitar-instructional convention they are called major-sixth vectors/shapes

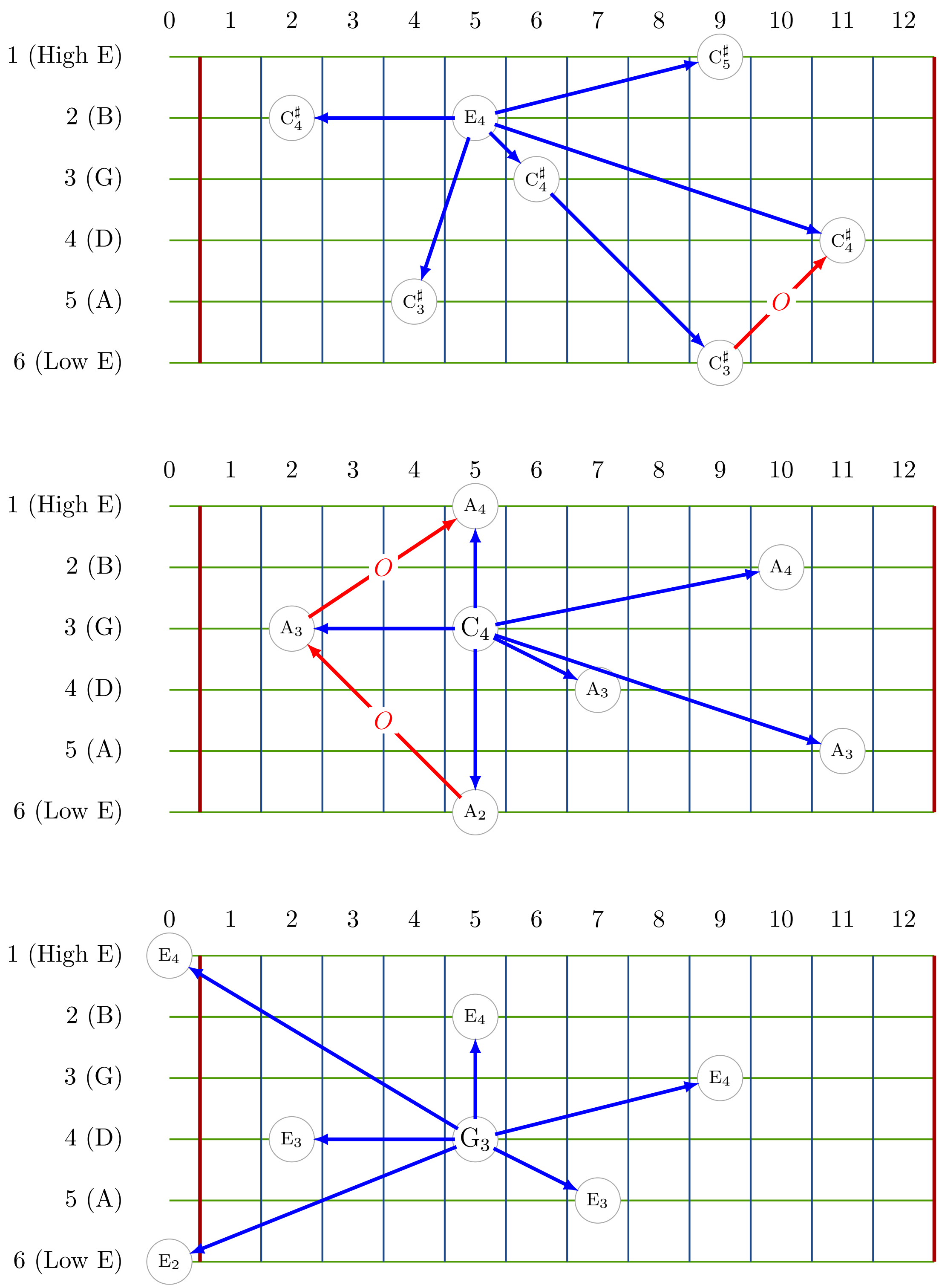


Figure 4 (continued): Major sixth vectors on the guitar fretboard
Blue = M_6 vector; Red = Octave vector

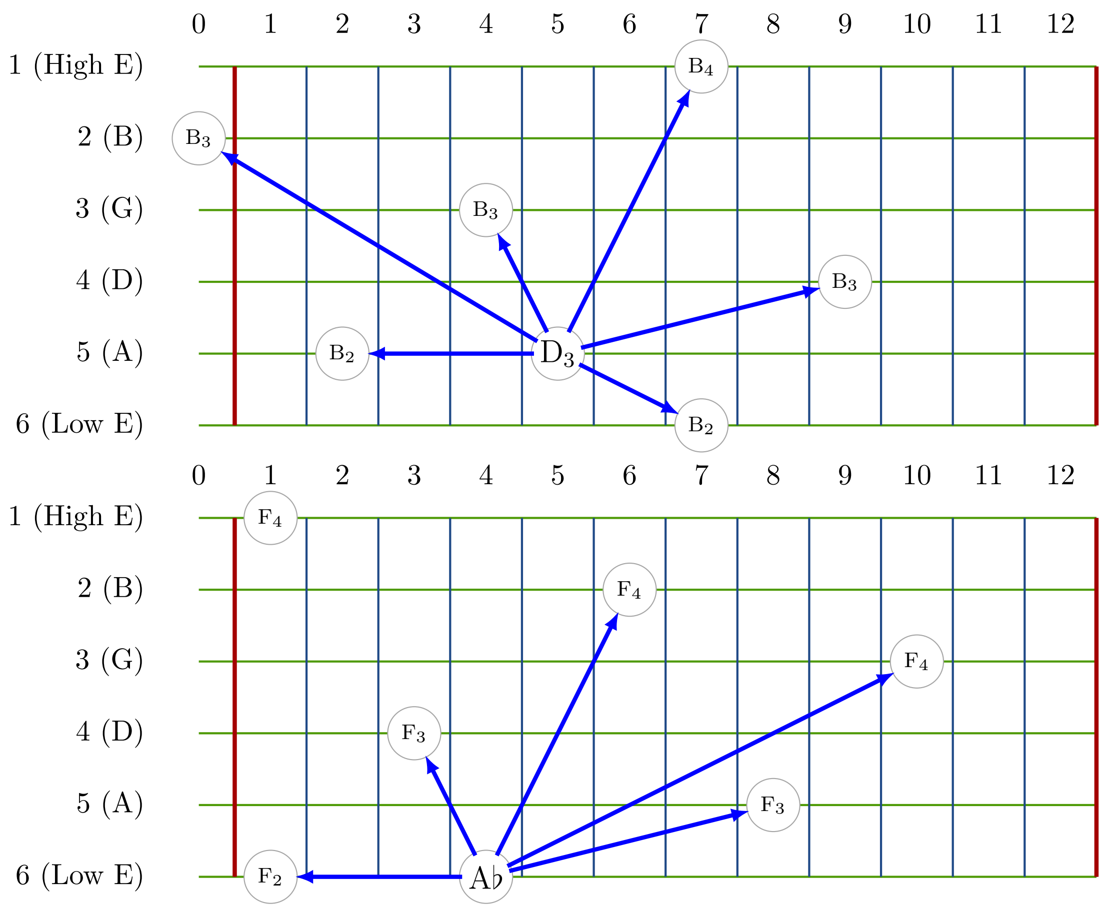


Figure 4 (continued): Major sixth vectors on the guitar fretboard
■ Blue = M_6 vector; ■ Red = Octave vector

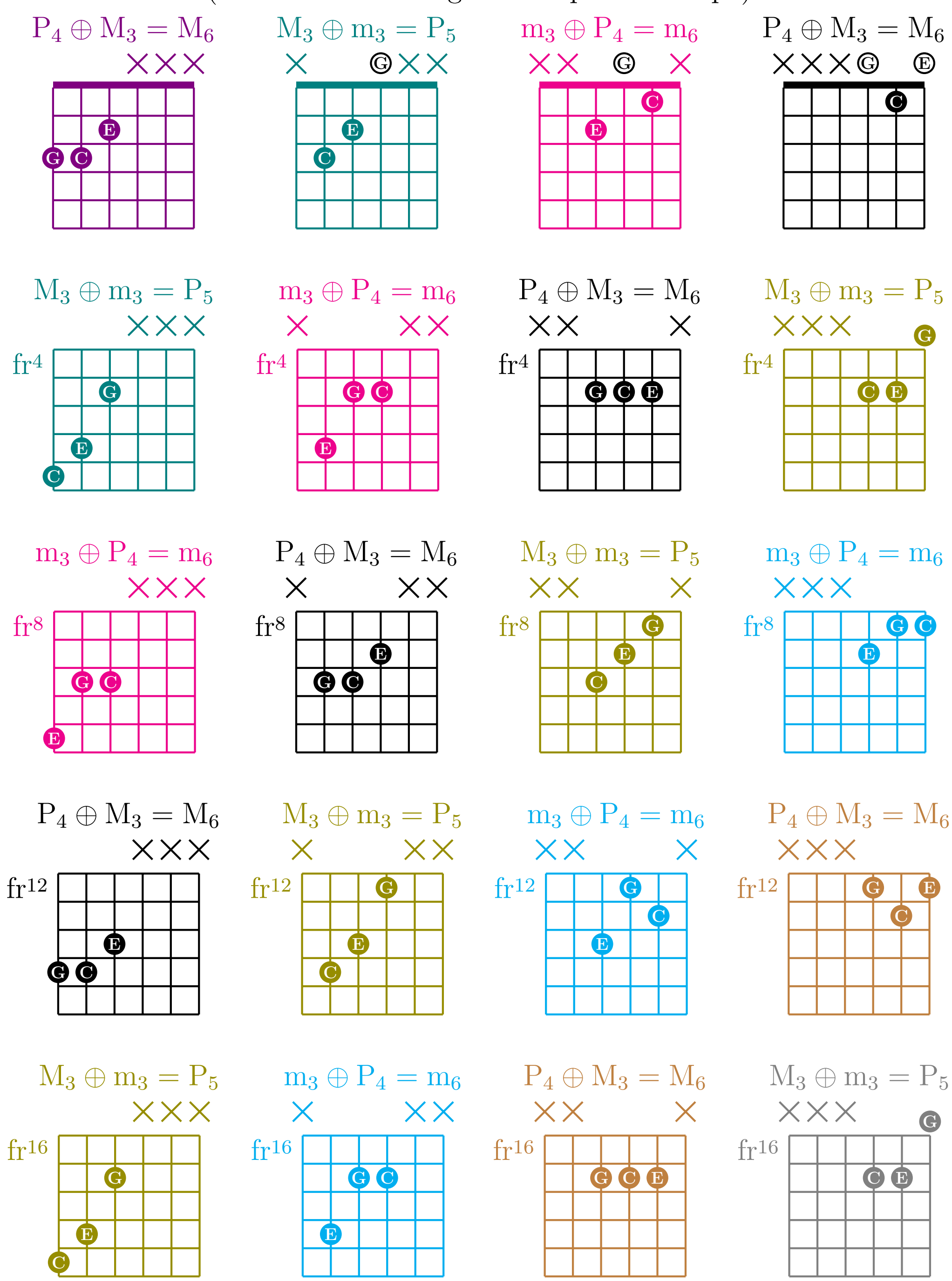
4.2. Pitch Equivalent Voicings

It is shown in ([18]) that there exists a permutation f of the set of 3-note voicings of the diatonic triads which preserves open/closed parity and rotates inversions. When lifted to $\mathbb{Z}$, the forward orbits of f are infinite ascending-pitch sequences of voicings. These orbits determine orders on both the open and closed triad voicings. Two chord voicings are said to be *pitch equivalent* if they have the same pitch in this order. This defines an equivalence relation ([15]) on the set of voicings.

Table (9) is a 5×4 array $X(i, j)$ of some commonly used voicings of the C closed triads on a 21 fret guitar showing their decomposition into intervals. The chords are arranged so that different colors correspond to different chord pitch equivalence classes. Chords on a given row fall in roughly the same range of frets. Each row contains a root, first inversion and second inversion triad of the closed C. The fourth chord is an upward pitch translation by one octave of the first chord. Thus, for example, $X(3,4)$ is one octave higher than $X(3,1)$ and of the same inversion/permutation.

Table 9: C major closed triads

Chord pitch is constant on fingerings of the same color
(located on the diagonals of positive slope).



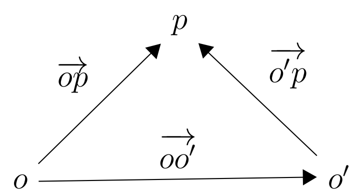
4.3. Coordinates

Let A be the M-torsor of equal temperament pitch classes over the module $M = \mathbb{Z}/12\mathbb{Z}$ acting freely and transitively on A by interval addition. A itself is not a group. However, choice of origin $o \in A$ (for example, the pitch class C) produces a canonical bijection $A \to M$ given by $p \mapsto p - o$. In music the pitch o is called the tonic. The inverse map $M \to A$ is $m \mapsto o + m$. In music the module element $op = p - o$ is called the *coordinate* of p relative to the origin o.

The choice $o =$ C, yields the standard coordinates *pitch class* $\{0, 1, 2, \ldots, 11\}$. A change of tonic is called a transposition and corresponds to a different choice of bijection $A \to M$. Since

$$\overrightarrow{op} = \overrightarrow{oo'} + \overrightarrow{o'p}$$

the new coordinates are just the old ones translated by a fixed element of M.



To transpose from C to G, take $o' = G$ and $o = C$, so that $CG = G - C = 7$. Then for all p

$$o'p = op - 7.$$

The new coordinates of the pitch classes are {5, 6, 7, 8, 9, 10, 11, 0, 1, 2, 3, 4}

If integer coordinates, e.g., MIDI numbers, of absolute pitches are required, the free module $\mathbb{Z}$ must be used in place of $\mathbb{Z}/12\mathbb{Z}$. In this case, fixing an origin now determines coordinates in $\mathbb{Z}$. Each coordinate $\bar{p} \in \mathbb{Z}$ is now a lift of some (not unique) pitch class coordinate $\bar{p} \in \mathbb{Z}/12\mathbb{Z}$. I.e., $\pi(\bar{p}) = p$ where $\pi : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/12\mathbb{Z}$ is the canonical projection. In this case transposing by an octave corresponds to adding multiples of 12..

Notice that if a,b and p are any 3 pitches, then

$$\overrightarrow{pa} - \overrightarrow{pb} = a - b$$

is independent of p. In particular, any two choices of origin differ by a fixed translation in M, so the coordinate systems are canonically related by addition of a constant interval. On the other hand, addition of pitches is physically meaningless. Different choices of tonic will lead to different values for a sum of pitches.

4.4. Subspaces

Definition 3. *Let A be an affine space over a module M. An affine subspace of A is a coset*

$$a + N = \{a + n : n \in N\}$$

where $a \in A$ and $N \subseteq M$ is a submodule (i.e., a subgroup closed under scalar multiplication by integers).

Remark. If $N = M$, the full dimensional case, the subspace is called *improper*. In this case, it follows by transitivity of the action that $a + N = M$, the entire module.

The submodules of the $\mathbb{Z}$-module $\mathbb{Z}/12\mathbb{Z}$ are just its additive subgroups. Since $\mathbb{Z}/12\mathbb{Z}$ is cyclic, and a cyclic group of order n has exactly one subgroup (hence one submodule) for each divisor of n there are exactly 6 submodules:

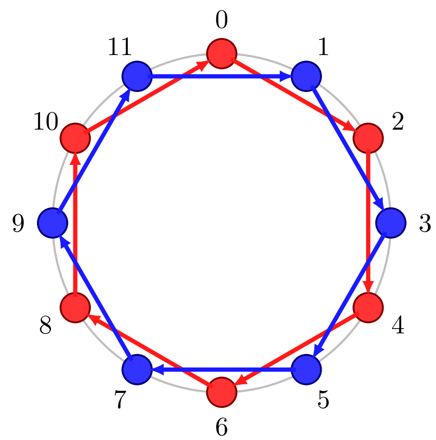
Table 10: Submodules of $\mathbb{Z}/12\mathbb{Z}$ and their musical interpretations

Submodule	**Elements**	**Cosets(subspaces) represent**
$\{0\}$	$\{0\}$	The 12 notes
$\langle 6\rangle \cong \mathbb{Z}/2\mathbb{Z}$	$\{0, 6\}$	The 6 tritones
$\langle 4\rangle \cong \mathbb{Z}/3\mathbb{Z}$	$\{0, 4, 8\}$	The 4 augmented triads
$\langle 3\rangle \cong \mathbb{Z}/4\mathbb{Z}$	$\{0, 3, 6, 9\}$	The 3 diminished-seventh chords
$\langle 2\rangle \cong \mathbb{Z}/6\mathbb{Z}$	$\{0, 2, 4, 6, 8, 10\}$	The two whole-tone scales
$\mathbb{Z}/12\mathbb{Z}$	S	The set of pitches S

By Lagrange's theorem, the number of cosets, of a subgroup H of a finite group G is just the ratio of the orders $|G|/|H|$. For example, the submodule $\langle 4\rangle = \{0, 4, 8\}$ determines 4 affine subspaces $p + \langle 4\rangle, \quad p \in A$. Using standard pitch numbering, $(0 = \text{C})$, these are precisely the four distinct augmented triads in 12-tone equal temperament:

1. C augmented $= \{0, 4, 8\} = 0 + \langle 4\rangle$
2. C♯ augmented $= \{1, 5, 9\} = 1 + \langle 4\rangle$
3. D augmented $= \{2, 6, 10\} = 2 + \langle 4\rangle$
4. D♯ augmented $= \{3, 7, 11\} = 3 + \langle 4\rangle$

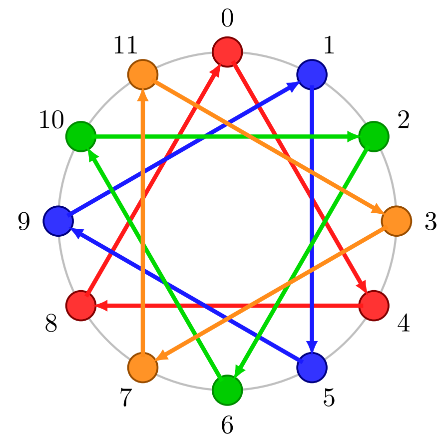
The various cosets corresponding to the submodules $\langle 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$, and $\langle 2\rangle$, are drawn in Figure (5) and their musical significance noted.



Key

- Red: Even coset $\{0, 2, 4, 6, 8, 10\}$ (whole-tone scale)
- Blue: Odd coset $\{1, 3, 5, 7, 9, 11\}$

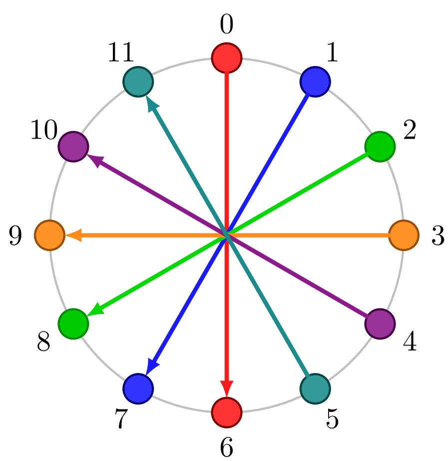
(a) The two cosets of $\langle 2 \rangle$ (even vs odd pitch classes)



Key

- Red: $\{0, 4, 8\}$ = C aug
- Blue: $\{1, 5, 9\}$ = C♯ aug
- Green: $\{2, 6, 10\}$ = D aug
- Orange: $\{3, 7, 11\}$ = E♭ aug

(b) The four cosets of $\langle 4 \rangle$ (the four augmented triads)

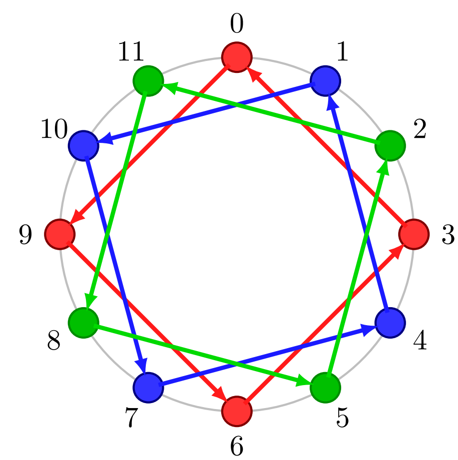


Key

- Red: $\{0, 6\}$ = C / F♯
- Blue: $\{1, 7\}$ = C♯ / G
- Green: $\{2, 8\}$ = D / G♯
- Orange: $\{3, 9\}$ = E♭ / A
- Purple: $\{4, 10\}$ = E / B♭
- Teal: $\{5, 11\}$ = F / B

Arrows = tritone (+6)

(c) The six cosets of $\langle 6 \rangle$ — the tritone dyads



Key

- Red: $\{0, 3, 6, 9\}$ = C°7 (major sixth orbit)
- Blue: $\{1, 4, 7, 10\}$ = C♯°7 (major sixth orbit)
- Green: $\{2, 5, 8, 11\}$ = D°7 (major sixth orbit)

Arrows = major sixth transposition (+9 semitones)

(d) The three cosets of $\langle 3 \rangle$ (diminished seventh chords)

Figure 5: The cosets of the cyclic subgroups $\langle 2 \rangle$, $\langle 4 \rangle$, $\langle 6 \rangle$ and $\langle 3 \rangle$ of $\mathbb{Z}/12\mathbb{Z}$.

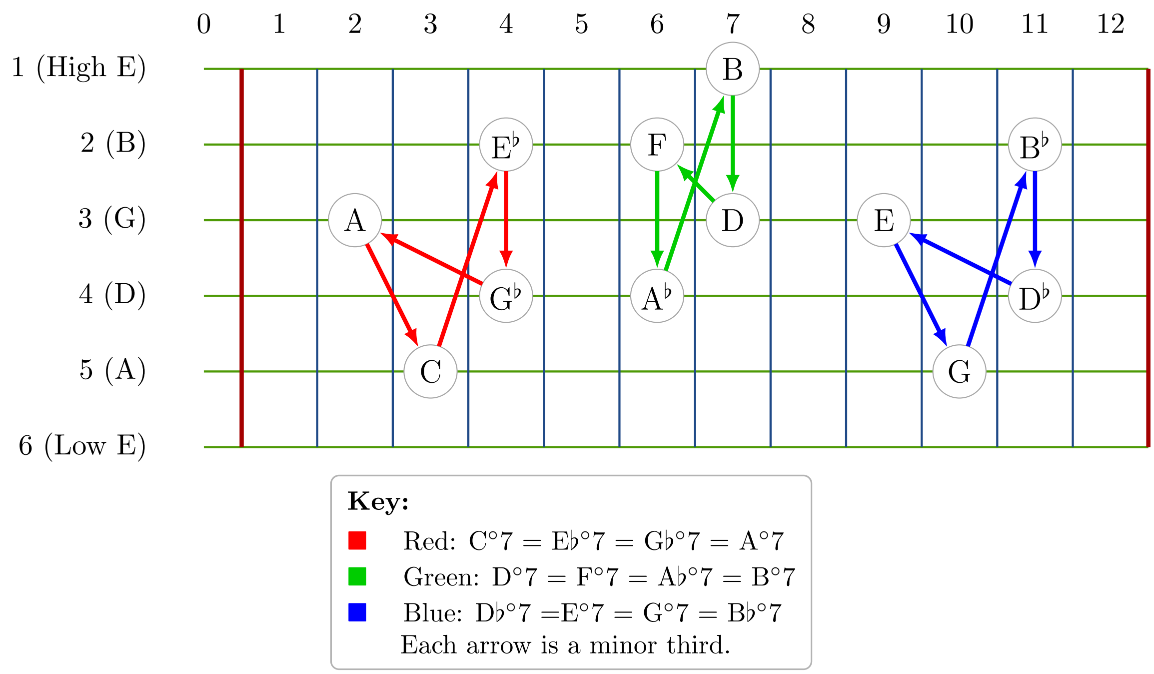


Figure 6: Voicings of the diminished 7 chords

Clearly, not every chord is an affine subspace. For example, B diminished

$$B^\circ = \{11 + \{0, 3, 6\}\}$$

is not a subspace because the set $\{0, 3, 6\}$ is not a submodule of $\mathbb{Z}/12\mathbb{Z}$. The actual submodule generated by 3, namely, $\langle 3\rangle = \{0, 3, 6, 9\}$ is the diminished seventh chord C, E♭, G♭, B♭♭) drawn in Figure (6).

5. Melody

5.1. Scales and Keys

A *scale* [13] is a finite sequence $s = (s_k)$ in the affine space of pitches A. Practical musical scales necessarily satisfy restrictions imposed by the physics of sound, the limitations of musical instruments, the range of the human ear, as well cultural and aesthetic preferences regarding the harmonies available to the scale. A *key* is an ordered pair (o, s) where o is origin for A and s is a scale. If (o, s) is a key, every term $s_k \in S$ can be written uniquely as

$$s_k = o + v_k, \qquad v_k \in M.$$

where the element v_k is the coordinate of s_k relative to the origin o of the key. Table(11) gives examples of some common keys.

Table 11: Examples of Keys as Ordered Pairs (o, s)

Key	Tonic o	Relative Vectors v_k	Scale Notes $s_k = o + v_k$
C major	C	$(0, 2, 4, 5, 7, 9, 11)$	$(0, 2, 4, 5, 7, 9, 11)$
G major	G	$(0, 2, 4, 5, 7, 9, 11)$	$(7, 9, 11, 0, 2, 4, 6)$
F major	F	$(0, 2, 4, 5, 7, 9, 11)$	$(5, 7, 9, 10, 0, 2, 4)$
C minor	C	$(0, 2, 3, 5, 7, 8, 10)$	$(0, 2, 3, 5, 7, 8, 10)$
A minor	A	$(0, 2, 3, 5, 7, 8, 10)$	$(9, 11, 0, 2, 4, 5, 7)$
C dorian	C	$(0, 2, 3, 5, 7, 9, 10)$	$(0, 2, 3, 5, 7, 9, 10)$
D dorian	D	$(0, 2, 3, 5, 7, 9, 10)$	$(2, 4, 5, 7, 9, 11, 0)$

The column v_k shows the relative coordinate vectors that define the mode; the column s_k shows the absolute pitch classes (mod 12) in ascending scale order.

The terms of a scale can be labelled in different ways. If L is a set of labels, and $\deg : S \to L$ a labelling, the label for s_k is called the kth scale degree. The term *solmization* when the scale degree labels are syllables. A scale degree thus, indicates the position of a note relative to the tonic of the current key.

Examples. The following are possible labelings (scale degrees) of the C major scale:

1. $\hat{1}, \hat{2}, \dots, \hat{7}$ – position labelling $\deg(o + v_k) := k$.
2. {C, D, E, F, G, A, B} – letter labelling
3. {I, II, III, IV, V, VI, VII} –Roman labelling
4. $\{\text{tonic}, \text{supertonic}, \text{mediant}, \text{subdominant}, \text{dominant}, \text{submediant}, \text{leadingtone.}\}$ – Traditional degree-role names[3]
5. $\{do, re, mi, fa, so, la, si\}$ the so called *solfège* solmization syllables[4].

[3] Warning. The label submediant is misleading since the submediant is not the counterpart of mediant in a manner consistent with the latin meanings of sub and super as the subdominant is the counterpart of dominant and the supertonic the counterpart of tonic. There is no reflection $x \mapsto c - x$ nor an affine map $x \mapsto ax + b$ from degree numbers to degree numbers which maps 3 to 6 (mediant to submediant) and also 5 to 4 (dominant to subdominant). Other types of maps are too permissive qualify as “natural counterpart maps”.

[4] The syllables (UT, RE, MI, FA, SOL, LA) in the original solfeggio scale begin phrases from the Gregorian chant hymn Ut Queant Laxis, written by Paulus Diaconus in the 8th century.

Example In C major the note B has coordinate 11 and scale degree 7 (leading tone).

D Dorian mode has the same set of pitches as C major. However, In this scale the note B now has coordinate 9, and scale degree 6. Table 12 gives the scale degree and coordinates of B in each of the classic white-key modes.

Table 12: White-key modes: same pitch collection, different tonics

Tonic	**Mode**	**Origin o**	**Coord. of B**	**Deg. of B**	**Scale coords**
C	Ionian	0	11	7	$0, 2, 4, 5, 7, 9, 11$
D	Dorian	2	9	6	$0, 2, 3, 5, 7, 9, 10$
E	Phrygian	4	7	5	$0, 1, 3, 5, 7, 8, 10$
F	Lydian	5	6	4	$0, 2, 4, 6, 7, 9, 11$
G	Mixolydian	7	4	3	$0, 2, 4, 5, 7, 9, 10$
A	Aeolian	9	2	2	$0, 2, 3, 5, 7, 8, 10$
B	Locrian	11	0	1	$0, 1, 3, 5, 6, 8, 10$

Let $M = \mathbb{Z}$ be the module of intervals in semitone units, and A is the affine space of pitches. A *melody* is a finite walk $p_0, p_1, \ldots, p_n$ in the affine space A. The vectors $d_k = p_k - p_{k-1} \in M$ for $k = 1, \ldots, n$ are called *melodic intervals.* The coordinate of a note p_k in the melody is the vector $v_k = p_k - p_0$. Notice that $v_k = \sum_{j=1}^{k} d_k = p_k - p_o$ In what follows a melodic interval d_k will be called a

1. *step* if $|d_k| \leq 2$
2. *skip* if $2 < |d_k| \leq 4$
3. *leap* if $|d_k| \geq 5$

5.2. Melodic Rules

Constraints on the sequence (d_k) are called *melodic rules.*

The rules of tonal melody are not absolute laws but preferences that can be expressed as conditional probabilities over the sequence (d_k) of scale-degree intervals. Because these intervals are independent of absolute pitch, the rules apply in any key.

The rules implicitly assume that music divides naturally into phrases[5] and the rules are to be applied locally to each phrase. Phrase boundaries mark structural

[5]This division is interpretive rather than purely objective relying on the presence of a variety of features, about which, most musicians agree. In computer music automatic boundary detection algorithms are used to detect phrase boundaries.

breaks, so intervals connecting two phrases are not treated as melodic and are excluded from an analysis of melodic form.

The following is a list of some of the most common melodic constraints.

1. **Conjunct motion.** Steps of a second and unisons/repeated notes predominate; probability falls rapidly with interval size.

$$P(|d_k| \le 2 \text{ semitones}) \gg P(|d_k| > 2).$$

2. **Leap compensation (opposite-direction resolution).** If $|d_{k-1}| \ge 3$,

$$P\big(\operatorname{sign}(d_k) = -\operatorname{sign}(d_{k-1})\big) \approx 1.$$

After a leap, the compensation is preferably by step.

3. **Tendency-tone resolution.** A scale degree s_t is a *tendency tone* when $P(s_{t+1} \mid s_t)$ is large. Tendency tones resolve by step (usually a semitone) to stable scale degrees, creating tension that seeks resolution.

   (a) Leading-tone resolution:

$$P(s_{t+1} = \hat{1} \mid s_t = \hat{7}) \approx 1.$$

   (b) Chordal-seventh resolution (in a seventh chord):

$$P(s_{t+1} = \hat{3} \mid s_t = \hat{4}) \quad \text{very high.}$$

   (c) Supertonic resolution:

$$P(s_{t+1} = \hat{1} \text{ or } \hat{3} \mid s_t = \hat{2}) \quad \text{elevated.}$$

   (d) Submediant resolution:

$$P(s_{t+1} = \hat{5} \mid s_t = \hat{6}) \quad \text{elevated.}$$

4. **Augmented (and diminished) intervals.**

$$P(d_k \text{ is an augmented interval}) \approx 0.$$

Diminished intervals, when they occur, are almost always followed by stepwise motion in the opposite direction.

5. **Consecutive leaps.** Consecutive leaps must outline a triad (three-note chord).

6. **Diatonic membership.**

   (a) $P(s_t \text{ is diatonic}) \gg P(s_t \text{ is chromatic})$.
   (b) $P(s_{t+1} \text{ is diatonic} \mid s_t \text{ is chromatic}) \approx 1$.

7. **Tonal hierarchy within the diatonic set.**

$$P(\hat{1}) > P(\hat{5}) > P(\hat{3}) \approx P(\hat{2}) > P(\hat{4}) \approx P(\hat{6}) > P(\hat{7}).$$

8. **Melodic contour.** The partial sums $s_m = \sum_{k=1}^{m} d_k$ (position relative to p_0) should exhibit a global maximum or minima (one clear climax), with smooth approach and retreat.

9. **Range.** All p_k lie within an affine interval of length at most roughly 12–19 semitones (a typical vocal range).

10. **Flow (repeated notes).** Repeated notes have moderate probability, but long strings of exact repetitions have low probability.

11. **Shape constraints.** The probability of a rising-then-falling (or falling-then-rising) contour is high; random-walk contours have low probability.

12. **Endpoint / cadential constraint.**
   (a) $P(\text{phrase ends on } \hat{1})$ is high (authentic or plagal cadence).
   (b) $P(\text{phrase ends on } \hat{5})$ is elevated in half-cadence contexts.

13. **Opening constraint.**

$$P(s_1 \in \{\hat{1}, \hat{3}, \hat{5}\}) \quad \text{high.}$$

5.3. Greensleeves

This section examines the degree to which the traditional song Greensleeves conforms to the above melodic rules. A simple guitar[6] score of the music can be found in Figure (7). The corresponding melodic intervals are plotted in Figure (8). Figure (9) plots the pitch coordinates relative to the tonic. For convenience, histograms of the interval class and pitch class distributions are plotted in Figure (10) and Figure (11). The chromatic notes ($D^\flat$, B) are treated as temporary alterations rather than scale degrees

[6] Note that sheet music for guitar is written one octave higher than it sounds.

5.3.1. Greensleeves Melody

From Figure (9) it is seen (after excluding those intervals which connect two phrases) that the melody has a two-part design each part containing two phrases – a so called *double period* design. Specifically:

1. Part 1 (phrases 1–2)
   (a) Both phrases are clear arches (convex) and open with the same rising figure from D4 to a maximum of B♭4 (8 semitones above the tonic) before descending.
   (b) Phrase 1 ends on A3 (half cadence).
   (c) Phrase 2 ends on D4 (authentic cadence).
2. Part 2 (the higher register phrases 3–4)
   (a) Both phrases descend from a global maximum pitch C5 with only a small secondary rises later.
   (b) Phrase 3 ends on A3 (half cadence).
   (c) Phrase 4 ends on D4 (authentic cadence).

Thus, phrases 1, 3 both dip to a global minimum pitch of A3. Stylistically, the lower pair of phrases establishes the main idea; the higher pair reinforces it. Frequent returns to the original pitch level (or a close neighbour) enhance the song-like quality. The higher register phrases (3-4) both begin on the global maximum pitch (midi C5) which is 10 semitones above the tonic, They repeat an arch type similar but of lower pitch than phrases 1,2. Phrases 1, 3 both dip to a global minimum pitch of the dominant (Midi A3). Stylistically, the lower pair of phrases establishes the main idea; the higher pair reinforces it. Frequent returns to the original pitch level (or a close neighbour) enhance the song-like quality.

Considering in turn each of the above rules for melodic intervals:

1. Conjunct motion. Steps/unisons predominate (≈ 75 % of intervals have $|d_k| \leq 2$). Leaps are mostly 3-4 semitones. As explained previously, larger intervals which straddle phrase boundaries are not counted.
2. Leap compensation. Mixed observance. After some leaps (e.g., –4 then +2) the next interval reverses direction and is often a step; other leaps continue in the same direction (e.g., $+3$ then $+2$, or -3 then -4).
3. Tendency-tone resolution. In the D-minor context: The submediant (B♭ = 6) regularly falls by step to the dominant (A= 5); (C= 7) frequently rises to the dominant (D= 1); chromatic (D♭ = ♭2) often resolves by step to (D=1).

4. Augmented/diminished intervals. No augmented intervals occur. The occasional diminished-sounding successions (e.g., E–D♭-A) are immediately followed by stepwise or opposite-direction motion.

5. Consecutive leaps. Whenever two leaps follow each other (always -3 then -4), the three notes outline a triad (G–E–C or E–D♭–A).

6. Diatonic membership. Most notes belong to D natural minor. The two chromatic pitches (D♭, B) are almost always followed by a diatonic note (exceptions are brief D♭–B–D♭ exchanges).

7. Tonal hierarchy. Pitch-class frequencies roughly follow the expected order: $\hat{1}$ (D) is most frequent, followed by $\hat{2}$ (E) and $\hat{5}$ (A); $\hat{7}$ (C) is unexpectedly high, while $\hat{6}$ (B♭) is rare.

8. Melodic contour. Each phrase shows one clear high-point (B♭4 or C5) near the beginning, approached by ascent and left by a long descent with mild secondary undulations.

9. Range. Overall span is A3–C5 (15 semitones), comfortably inside a typical vocal range.

10. Flow (repeated notes). Unisons appear in short groups (two or three D's or C's); no extended repetition strings occur.

11. Shape constraints. All four phrases are clear arches or descents. Random-walk contours are absent.

12. Endpoint / cadential constraint. Phrases end alternately on A3 $= \hat{5}$ and D4 $= \hat{1}$ producing half-cadence and authentic-cadence effects.

13. Opening constraint. The first two phrases open on Dm$= \hat{1}$; the last two open on C$= \hat{7}$, only partially satisfying the preferred $\hat{1}$, $\hat{3}$, $\hat{5}$ set.

Figure 7: Greensleeves (traditional)

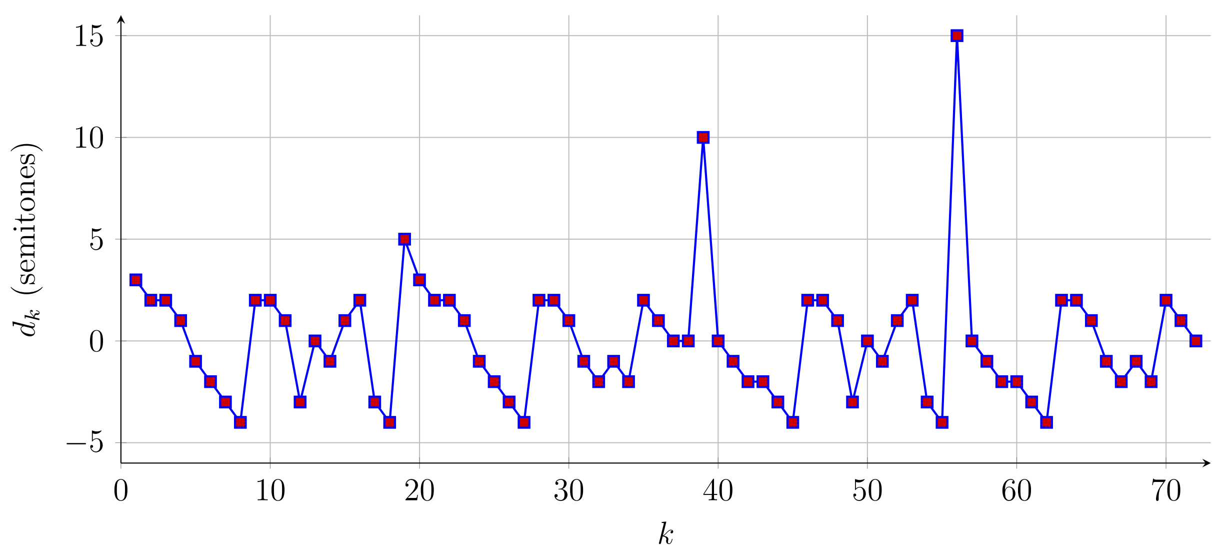


Figure 8: Greensleeves - Melodic Intervals

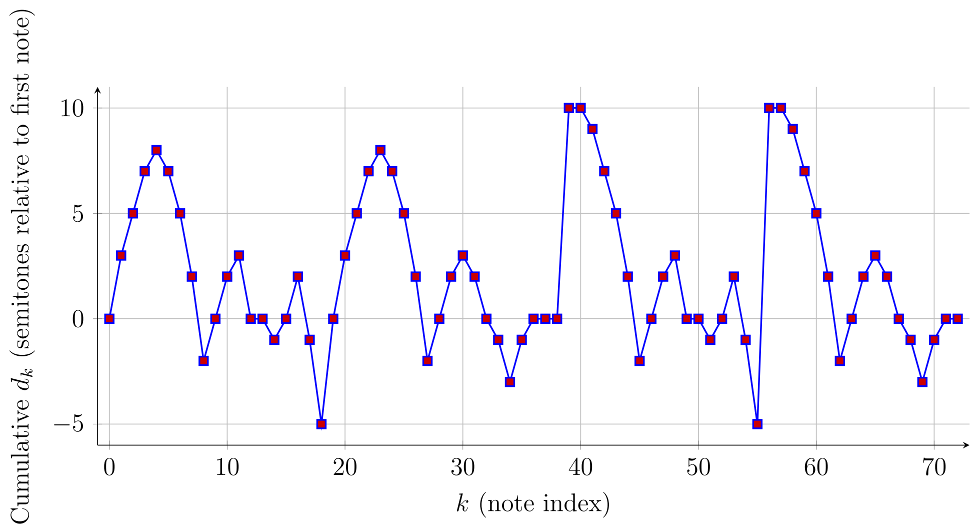


Figure 9: Greensleeves - Semitones relative to tonic

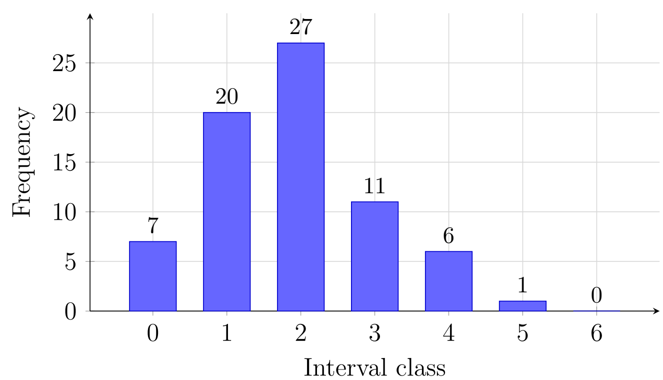


Figure 10: Greensleeves - Interval Class Distribution

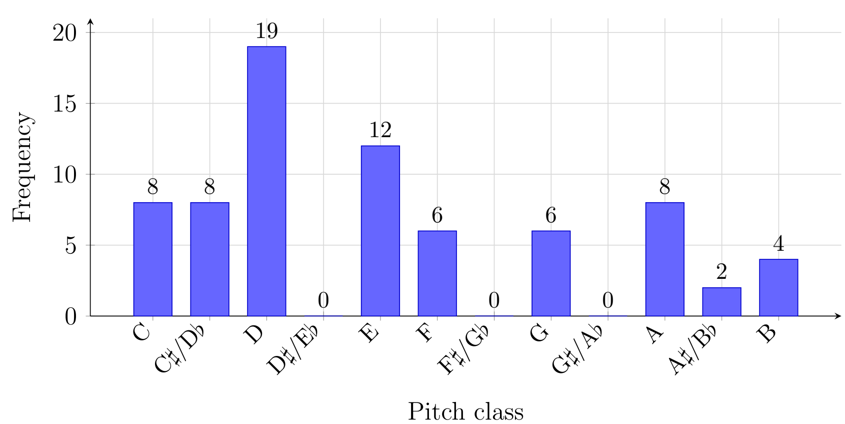


Figure 11: Greensleeves - Pitch Class Distribution

5.3.2. Greensleeves Harmony

Harmonic constraints in traditional Western music include assumptions that dominant harmonies (V and vii) tend to tonics (I), predominants (IV and ii) move to dominants, that root motion by descending fifth is especially favored, and so on [20]. The chord transitions in each phrase of Greensleeves are drawn as a graph in Figure (14). The chordal structure, as expected, strongly parallels the melody dividing into two parts, each of two phrases. The first phrase of each part is transitional terminating on the dominant V_7. The second phrase of each mirrors the first but ends decisively in a cadence.

The (i-III) transitions in the first half of the song serve to prolong the tonic (notice that III having two notes in common with i.) The moves from III to VI in the phrases 1 and 3 provide a darker, more modal feel, before a return to the dominant V7. Phrase 2, i–III–VII–v–i–V7–i, is the only one providing a full tonic-to-dominant-to-tonic motion, ending with a conclusive authentic cadence.

Part II phrases begins away from tonic with III–VII–v–i, providing a feeling of continuation rather than a fresh start. The V7–i close in phrase 4 gives strong finality to the piece.

In a study [22] of traditional Western music Dmitri Tymoczko assembled databases containing Roman numeral analyses of 56 Mozart piano-sonata movements and 70 Bach chorales [7]. The following are very approximate probabilities derived from the Bach/Mozart natural-minor data by re-normalizing to accommodate 5, the major triad on the dominant (A major when tonic is Dm).

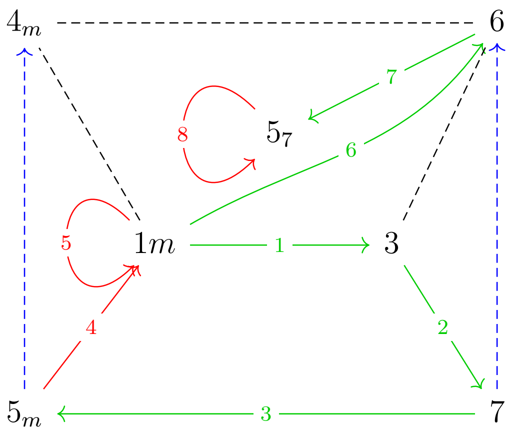
As expected in tonal minor, $5 = V$ becomes the strongest attractor from the predominants (2°, 4m) and from the tonic as expected in tonal minor. From 5 the overwhelming preference is back to 1m (authentic cadence). The probabilities from this table are color coded into the chord transitions in Figure (14).

from\to	1m	2°	3	4m	5m	6	7	5
1m	0.000	0.120	0.005	0.130	0.050	0.060	0.100	0.535
2°	0.015	0.000	0.015	0.000	0.050	0.000	0.120	0.800
3	0.030	0.030	0.000	0.600	0.040	0.150	0.060	0.090
4m	0.280	0.100	0.000	0.000	0.040	0.000	0.100	0.480
5m	0.450	0.000	0.020	0.100	0.000	0.150	0.080	0.200
6	0.030	0.520	0.015	0.150	0.040	0.000	0.095	0.150
7	0.650	0.005	0.005	0.020	0.030	0.000	0.000	0.290
5	0.820	0.000	0.010	0.040	0.020	0.070	0.030	0.000

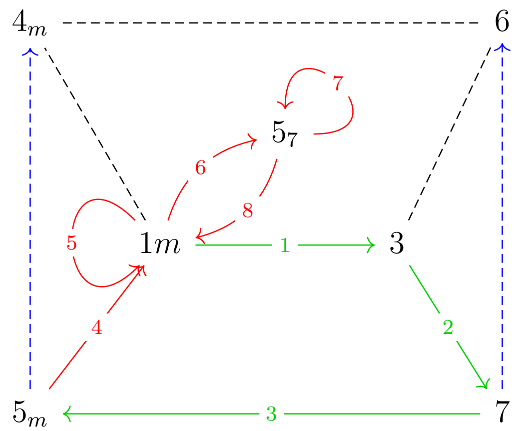
Figure 12: Melodic Minor Chord Transition Probabilities

i	ii°	III	iv	v	VI	VII	V
1m	2°	3	4m	5m	6	7	5
Dm	E°	F major	Gm	Am	B♭ major	C major	A major

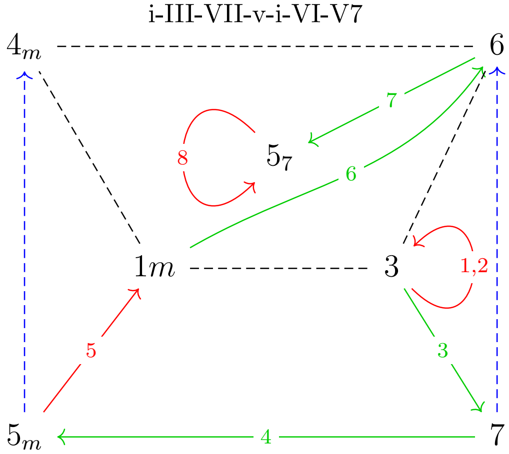
Figure 13: Nashville and Roman Numbering for Dm Chords



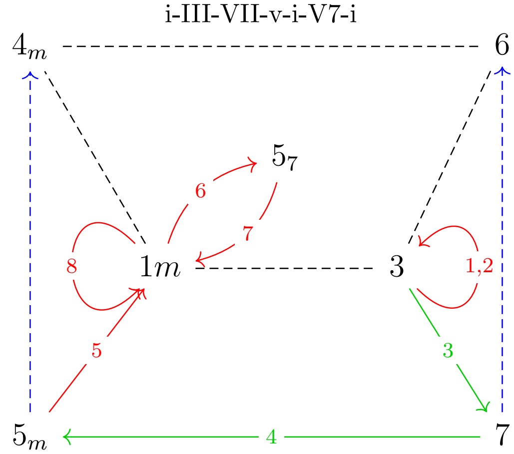
(a) Part I Phrase 1
i-III-VII-v-i-VI-V7



(b) Part I Phrase 2
i-III-VII-v-i-V7-i



(c) Part II Phrase 3
III-VII-v-i-VI-V7



(d) Part II Phrase 4
III-VII-v-i- V7-i

Key:

- Red: High $P \geq 0.2 - 0.3$
- $0.05 < P < 0.2$
- Blue: $P \leq 0.05$

Edge labels: Chord transition order
Solid edges: Transitions taken by the walk
Dashed edges: Transition avoided by walk

Figure 14: Greensleeves Chord Transitions

5.4. Further Reading

These introductory notes have focused on the mathematics pertaining to a single melody. In multipart music, harmonic constraints arise from considerations of *counterpoint*, i.e., the desire to connect the notes in a adjacent chords so as to form simultaneous melodies which combine together harmoniously. The mathematical models of counterpoint of can be found in [21].

An early reference which explicitly describes the chromatic set of notes as a torsor is John Baez [3]. His extensive series [2] on the mathematics of tuning systems discusses just intonation, equal temperament, the Tonnetz, and related musical topics.

David Lewin [9], is the standard reference for transformational music theory. His notion of a *generalized interval system* (GIS) is essentially a torsor.

Guerino Mazzola et al., The Topos of Music (2002 and later volumes) [11]— use $\mathbb{Z}$-modules (and modules over general rings) extensively for pitch spaces.

Dmitri Tymoczko's geometric music theory (e.g., A Geometry of Music, papers on "musical vectors"): Treats pitch/chord spaces as affine spaces (or orbifolds thereof) where intervals are vectors and transposition is translation.

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7. Appendix - Tables of Chord Constructions

Table 13: Augmented Triad Constructions

Chord	Example	Construction
closed root position	1-3-#5	$M_3 \oplus M_3 = A_5$
closed first inversion	3-#5-1	$M_3 \oplus M_3 = A_5$
closed second inversion	#5-1-3	$M_3 \oplus M_3 = A_5$
open root position	1-#5-3	$A_5 \oplus A_5 = M_{10}$
open first inversion	3-1-#5	$A_5 \oplus A_5 = M_{10}$
open second inversion	#5-3-1	$A_5 \oplus A_5 = M_{10}$

Table 14: Suspended Fourth Triad Constructions

Chord	Example	Construction
closed root position	1-4-5	$P_4 \oplus M_2 = P_5$
closed first inversion	4-5-1	$M_2 \oplus P_4 = P_5$
closed second inversion	5-1-4	$P_4 \oplus P_4 = m_7$
open root position	1-5-4	$P_5 \oplus m_7 = P_{11}$
open first inversion	4-1-5	$P_5 \oplus P_5 = M_9$
open second inversion	5-4-1	$m_7 \oplus P_5 = P_{11}$

Table 15: Suspended Second Triad Constructions

Chord	Example	Construction
closed root position	1-2-5	$M_2 \oplus P_4 = P_5$
closed first inversion	2-5-1	$P_4 \oplus P_4 = m_7$
closed second inversion	5-1-2	$P_4 \oplus M_2 = P_5$
open root position	1-5-2	$P_5 \oplus P_5 = M_9$
open first inversion	2-1-5	$m_7 \oplus P_5 = P_{11}$
open second inversion	5-2-1	$P_5 \oplus m_7 = P_{11}$

Table 16: Major Seventh Constructions

Chord	Example	Construction
closed root position	1-3-5-7	$M_3 \oplus m_3 \oplus M_3 = M_7$
closed first inversion	3-5-7-1	$m_3 \oplus M_3 \oplus m_2 = m_6$
closed second inversion	5-7-1-3	$M_3 \oplus m_2 \oplus M_3 = M_6$
closed third inversion	7-1-3-5	$m_2 \oplus M_3 \oplus m_3 = m_6$
open root position	1-5-7-3	$P_5 \oplus M_3 \oplus P_4 = M_{10}$
open first inversion	3-7-1-5	$P_5 \oplus m_2 \oplus P_5 = m_{10}$
open second inversion	5-1-3-7	$P_4 \oplus M_3 \oplus P_5 = M_{10}$
open third inversion	7-3-5-1	$P_4 \oplus m_3 \oplus P_4 = m_9$

Table 17: Dominant Seventh Constructions

Chord	Example	Construction
closed root position	1-3-5-b7	$M_3 \oplus m_3 \oplus m_3 = m_7$
closed first inversion	3-5-b7-1	$m_3 \oplus m_3 \oplus M_2 = m_6$
closed second inversion	5-b7-1-3	$m_3 \oplus M_2 \oplus M_3 = M_6$
closed third inversion	b7-1-3-5	$M_2 \oplus M_3 \oplus m_3 = M_6$
open root position	1-5-b7-3	$P_5 \oplus m_3 \oplus d_5 = M_{10}$
open first inversion	3-b7-1-5	$d_5 \oplus M_2 \oplus P_5 = m_{10}$
open second inversion	5-1-3-b7	$P_4 \oplus M_3 \oplus d_5 = m_{10}$
open third inversion	b7-3-5-1	$d_5 \oplus m_3 \oplus P_4 = M_9$

Table 18: Minor Seventh Constructions

Chord	Example	Construction
closed root position	1-b3-5-b7	$m_3 \oplus M_3 \oplus m_3 = m_7$
closed first inversion	b3-5-b7-1	$M_3 \oplus m_3 \oplus M_2 = M_6$
closed second inversion	5-b7-1-b3	$m_3 \oplus M_2 \oplus m_3 = m_6$
closed third inversion	b7-1-b3-5	$M_2 \oplus m_3 \oplus M_3 = M_6$
open root position	1-5-b7-b3	$P_5 \oplus m_3 \oplus P_4 = m_{10}$
open first inversion	b3-b7-1-5	$P_5 \oplus M_2 \oplus P_5 = M_{10}$
open second inversion	5-1-b3-b7	$P_4 \oplus m_3 \oplus P_5 = m_{10}$
open third inversion	b7-b3-5-1	$P_4 \oplus M_3 \oplus P_4 = M_9$

Table 19: Minor Major Seventh Constructions

Chord	Example	Construction
closed root position	1-b3-5-7	$m_3 \oplus M_3 \oplus M_3 = M_7$
closed first inversion	b3-5-7-1	$M_3 \oplus M_3 \oplus m_2 = M_6$
closed second inversion	5-7-1-b3	$M_3 \oplus m_2 \oplus m_3 = m_6$
closed third inversion	7-1-b3-5	$m_2 \oplus m_3 \oplus M_3 = m_6$
open root position	1-5-7-b3	$P_5 \oplus M_3 \oplus M_3 = m_{10}$
open first inversion	b3-7-1-5	$m_6 \oplus m_2 \oplus P_5 = M_{10}$
open second inversion	5-1-b3-7	$P_4 \oplus m_3 \oplus m_6 = M_{10}$
open third inversion	7-b3-5-1	$M_3 \oplus M_3 \oplus P_4 = m_9$

Table 20: Half-Diminished Seventh Constructions

Chord	Example	Construction
closed root position	1-b3-b5-b7	$m_3 \oplus m_3 \oplus M_3 = m_7$
closed first inversion	b3-b5-b7-1	$m_3 \oplus M_3 \oplus M_2 = M_6$
closed second inversion	b5-b7-1-b3	$M_3 \oplus M_2 \oplus m_3 = M_6$
closed third inversion	b7-1-b3-b5	$M_2 \oplus m_3 \oplus m_3 = m_6$
open root position	1-b5-b7-b3	$d_5 \oplus M_3 \oplus P_4 = m_{10}$
open first inversion	b3-b7-1-b5	$P_5 \oplus M_2 \oplus d_5 = m_{10}$
open second inversion	b5-1-b3-b7	$d_5 \oplus m_3 \oplus P_5 = M_{10}$
open third inversion	b7-b3-b5-1	$P_4 \oplus m_3 \oplus d_5 = M_9$

Table 21: Diminished Seventh Constructions

Chord	Example	Construction
closed root position	1-b3-b5-bb7	$m_3 \oplus m_3 \oplus m_3 = M_6$
closed first inversion	b3-b5-bb7-1	$m_3 \oplus m_3 \oplus m_3 = M_6$
closed second inversion	b5-bb7-1-b3	$m_3 \oplus m_3 \oplus m_3 = M_6$
closed third inversion	bb7-1-b3-b5	$m_3 \oplus m_3 \oplus m_3 = M_6$
open root position	1-b5-bb7-b3	$d_5 \oplus m_3 \oplus d_5 = m_{10}$
open first inversion	b3-bb7-1-b5	$d_5 \oplus m_3 \oplus d_5 = m_{10}$
open second inversion	b5-1-b3-bb7	$d_5 \oplus m_3 \oplus d_5 = m_{10}$
open third inversion	bb7-b3-b5-1	$d_5 \oplus m_3 \oplus d_5 = m_{10}$

Table 22: Augmented Major Seventh Constructions

Chord	Example	Construction
closed root position	1-3-#5-7	$M_3 \oplus M_3 \oplus m_3 = M_7$
closed first inversion	3-#5-7-1	$M_3 \oplus m_3 \oplus m_2 = m_6$
closed second inversion	#5-7-1-3	$m_3 \oplus m_2 \oplus M_3 = m_6$
closed third inversion	7-1-3-#5	$m_2 \oplus M_3 \oplus M_3 = M_6$
open root position	1-#5-7-3	$A_5 \oplus m_3 \oplus P_4 = M_{10}$
open first inversion	3-7-1-#5	$P_5 \oplus m_2 \oplus A_5 = M_{10}$
open second inversion	#5-1-3-7	$M_3 \oplus M_3 \oplus P_5 = m_{10}$
open third inversion	7-3-#5-1	$P_4 \oplus M_3 \oplus M_3 = m_9$

Table 23: Augmented Dominant Seventh Constructions

Chord	Example	Construction
closed root position	1-3-#5-b7	$M_3 \oplus M_3 \oplus M_2 = m_7$
closed first inversion	3-#5-b7-1	$M_3 \oplus M_2 \oplus M_2 = m_6$
closed second inversion	#5-b7-1-3	$M_2 \oplus M_2 \oplus M_3 = m_6$
closed third inversion	b7-1-3-#5	$M_2 \oplus M_3 \oplus M_3 = m_7$
open root position	1-#5-b7-3	$A_5 \oplus M_2 \oplus d_5 = M_{10}$
open first inversion	3-b7-1-#5	$d_5 \oplus M_2 \oplus A_5 = M_{10}$
open second inversion	#5-1-3-b7	$M_3 \oplus M_3 \oplus d_5 = M_9$
open third inversion	b7-3-#5-1	$d_5 \oplus M_3 \oplus M_3 = M_9$

Table 24: Compound Diad Constructions

Interval	Name	Semitones	Simple Equivalent
M_{10}	Major Tenth	16	$M_3 \oplus O$
m_{10}	Minor Tenth	15	$m_3 \oplus O$
P_{11}	Perfect Eleventh	17	$P_4 \oplus O$
M_9	Major Ninth	14	$M_2 \oplus O$
A_{11}	Augmented Eleventh	18	$A_4 \oplus O$

Table 25: Major Ninth Constructions

Chord	Example	Construction
closed root position	1-3-5-7-9	$M_3 \oplus m_3 \oplus M_3 \oplus m_3 = M_9$
closed first inversion	3-5-7-9-1	$m_3 \oplus M_3 \oplus m_3 \oplus m_7 = m_{13}$
closed second inversion	5-7-9-1-3	$M_3 \oplus m_3 \oplus m_7 \oplus M_3 = M_{13}$
closed third inversion	7-9-1-3-5	$m_3 \oplus m_7 \oplus M_3 \oplus m_3 = m_{13}$
closed fourth inversion	9-1-3-5-7	$m_7 \oplus M_3 \oplus m_3 \oplus M_3 = M_{13}$
open root position	1-5-7-9-3	$P_5 \oplus M_3 \oplus m_3 \oplus M_2 = M_{10}$
open first inversion	3-7-9-1-5	$P_5 \oplus m_3 \oplus m_7 \oplus P_5 = m_{17}$
open second inversion	5-9-1-3-7	$P_5 \oplus m_7 \oplus M_3 \oplus P_5 = M_{17}$
open third inversion	7-1-3-5-9	$m_2 \oplus M_3 \oplus m_3 \oplus P_5 = m_{10}$
open fourth inversion	9-3-5-7-1	$M_2 \oplus m_3 \oplus M_3 \oplus m_2 = m_7$

Table 26: Dominant Ninth Constructions

Chord	Example	Construction
closed root position	1-3-5-b7-9	$M_3 \oplus m_3 \oplus m_3 \oplus M_3 = M_9$
closed first inversion	3-5-b7-9-1	$m_3 \oplus m_3 \oplus M_3 \oplus m_7 = m_{13}$
closed second inversion	5-b7-9-1-3	$m_3 \oplus M_3 \oplus m_7 \oplus M_3 = M_{13}$
closed third inversion	b7-9-1-3-5	$M_3 \oplus m_7 \oplus M_3 \oplus m_3 = M_{13}$
closed fourth inversion	9-1-3-5-b7	$m_7 \oplus M_3 \oplus m_3 \oplus m_3 = m_{13}$
open root position	1-5-b7-9-3	$P_5 \oplus m_3 \oplus M_3 \oplus M_2 = M_{10}$
open first inversion	3-b7-9-1-5	$d_5 \oplus M_3 \oplus m_7 \oplus P_5 = m_{17}$
open second inversion	5-9-1-3-b7	$P_5 \oplus m_7 \oplus M_3 \oplus d_5 = m_{17}$
open third inversion	b7-1-3-5-9	$M_2 \oplus M_3 \oplus m_3 \oplus P_5 = M_{10}$
open fourth inversion	9-3-5-b7-1	$M_2 \oplus m_3 \oplus m_3 \oplus M_2 = m_7$

Table 27: Minor Ninth Constructions

Chord	Example	Construction
closed root position	1-b3-5-b7-9	$m_3 \oplus M_3 \oplus m_3 \oplus M_3 = M_9$
closed first inversion	b3-5-b7-9-1	$M_3 \oplus m_3 \oplus M_3 \oplus m_7 = M_{13}$
closed second inversion	5-b7-9-1-b3	$m_3 \oplus M_3 \oplus m_7 \oplus m_3 = m_{13}$
closed third inversion	b7-9-1-b3-5	$M_3 \oplus m_7 \oplus m_3 \oplus M_3 = M_{13}$
closed fourth inversion	9-1-b3-5-b7	$m_7 \oplus m_3 \oplus M_3 \oplus m_3 = m_{13}$
open root position	1-5-b7-9-b3	$P_5 \oplus m_3 \oplus M_3 \oplus m_2 = m_{10}$
open first inversion	b3-b7-9-1-5	$P_5 \oplus M_3 \oplus m_7 \oplus P_5 = M_{17}$
open second inversion	5-9-1-b3-b7	$P_5 \oplus m_7 \oplus m_3 \oplus P_5 = m_{17}$
open third inversion	b7-1-b3-5-9	$M_2 \oplus m_3 \oplus M_3 \oplus P_5 = M_{10}$
open fourth inversion	9-b3-5-b7-1	$m_2 \oplus M_3 \oplus m_3 \oplus M_2 = m_7$

Table 28: Minor Major Ninth Constructions

Chord	Example	Construction
closed root position	1-b3-5-7-9	$m_3 \oplus M_3 \oplus M_3 \oplus m_3 = M_9$
closed first inversion	b3-5-7-9-1	$M_3 \oplus M_3 \oplus m_3 \oplus m_7 = M_{13}$
closed second inversion	5-7-9-1-b3	$M_3 \oplus m_3 \oplus m_7 \oplus m_3 = m_{13}$
closed third inversion	7-9-1-b3-5	$m_3 \oplus m_7 \oplus m_3 \oplus M_3 = m_{13}$
closed fourth inversion	9-1-b3-5-7	$m_7 \oplus m_3 \oplus M_3 \oplus M_3 = M_{13}$
open root position	1-5-7-9-b3	$P_5 \oplus M_3 \oplus m_3 \oplus m_2 = m_{10}$
open first inversion	b3-7-9-1-5	$m_6 \oplus m_3 \oplus m_7 \oplus P_5 = M_{17}$
open second inversion	5-9-1-b3-7	$P_5 \oplus m_7 \oplus m_3 \oplus m_6 = M_{17}$
open third inversion	7-1-b3-5-9	$m_2 \oplus m_3 \oplus M_3 \oplus P_5 = m_{10}$
open fourth inversion	9-b3-5-7-1	$m_2 \oplus M_3 \oplus M_3 \oplus m_2 = m_7$

Table 29: Half-Diminished Ninth Constructions

Chord	Example	Construction
closed root position	1-b3-b5-b7-9	$m_3 \oplus m_3 \oplus M_3 \oplus M_3 = M_9$
closed first inversion	b3-b5-b7-9-1	$m_3 \oplus M_3 \oplus M_3 \oplus m_7 = M_{13}$
closed second inversion	b5-b7-9-1-b3	$M_3 \oplus M_3 \oplus m_7 \oplus m_3 = M_{13}$
closed third inversion	b7-9-1-b3-b5	$M_3 \oplus m_7 \oplus m_3 \oplus m_3 = m_{13}$
closed fourth inversion	9-1-b3-b5-b7	$m_7 \oplus m_3 \oplus m_3 \oplus M_3 = m_{13}$
open root position	1-b5-b7-9-b3	$d_5 \oplus M_3 \oplus M_3 \oplus m_2 = m_{10}$
open first inversion	b3-b7-9-1-b5	$P_5 \oplus M_3 \oplus m_7 \oplus d_5 = m_{17}$
open second inversion	b5-9-1-b3-b7	$m_6 \oplus m_7 \oplus m_3 \oplus P_5 = M_{17}$
open third inversion	b7-1-b3-b5-9	$M_2 \oplus m_3 \oplus m_3 \oplus m_6 = M_{10}$
open fourth inversion	9-b3-b5-b7-1	$m_2 \oplus m_3 \oplus M_3 \oplus M_2 = m_7$

Table 30: Diminished Ninth Constructions

Chord	Example	Construction
closed root position	1-b3-b5-bb7-b9	$m_3 \oplus m_3 \oplus m_3 \oplus M_3 = m_9$
closed first inversion	b3-b5-bb7-b9-1	$m_3 \oplus m_3 \oplus M_3 \oplus M_7 = M_{13}$
closed second inversion	b5-bb7-b9-1-b3	$m_3 \oplus M_3 \oplus M_7 \oplus m_3 = M_{13}$
closed third inversion	bb7-b9-1-b3-b5	$M_3 \oplus M_7 \oplus m_3 \oplus m_3 = M_{13}$
closed fourth inversion	b9-1-b3-b5-bb7	$M_7 \oplus m_3 \oplus m_3 \oplus m_3 = m_{13}$
open root position	1-b5-bb7-b9-b3	$d_5 \oplus m_3 \oplus M_3 \oplus M_2 = m_{10}$
open first inversion	b3-bb7-b9-1-b5	$d_5 \oplus M_3 \oplus M_7 \oplus d_5 = m_{17}$
open second inversion	b5-b9-1-b3-bb7	$P_5 \oplus M_7 \oplus m_3 \oplus d_5 = m_{17}$
open third inversion	bb7-1-b3-b5-b9	$m_3 \oplus m_3 \oplus m_3 \oplus P_5 = M_{10}$
open fourth inversion	b9-b3-b5-bb7-1	$M_2 \oplus m_3 \oplus m_3 \oplus m_3 = M_7$

Table 31: Augmented Ninth Constructions

Chord	Example	Construction
closed root position	1-3-#5-b7-9	$M_3 \oplus M_3 \oplus M_2 \oplus M_3 = M_9$
closed first inversion	3-#5-b7-9-1	$M_3 \oplus M_2 \oplus M_3 \oplus m_7 = m_{13}$
closed second inversion	#5-b7-9-1-3	$M_2 \oplus M_3 \oplus m_7 \oplus M_3 = m_{13}$
closed third inversion	b7-9-1-3-#5	$M_3 \oplus m_7 \oplus M_3 \oplus M_3 = m_{14}$
closed fourth inversion	9-1-3-#5-b7	$m_7 \oplus M_3 \oplus M_3 \oplus M_2 = m_{13}$
open root position	1-#5-b7-9-3	$A_5 \oplus M_2 \oplus M_3 \oplus M_2 = M_{10}$
open first inversion	3-b7-9-1-#5	$d_5 \oplus M_3 \oplus m_7 \oplus A_5 = M_{17}$
open second inversion	#5-9-1-3-b7	$d_5 \oplus m_7 \oplus M_3 \oplus d_5 = M_{16}$
open third inversion	b7-1-3-#5-9	$M_2 \oplus M_3 \oplus M_3 \oplus d_5 = M_{10}$
open fourth inversion	9-3-#5-b7-1	$M_2 \oplus M_3 \oplus M_2 \oplus M_2 = m_7$

Table 32: Augmented Major Ninth Constructions

Chord	Example	Construction
closed root position	1-3-#5-7-9	$M_3 \oplus M_3 \oplus m_3 \oplus m_3 = M_9$
closed first inversion	3-#5-7-9-1	$M_3 \oplus m_3 \oplus m_3 \oplus m_7 = m_{13}$
closed second inversion	#5-7-9-1-3	$m_3 \oplus m_3 \oplus m_7 \oplus M_3 = m_{13}$
closed third inversion	7-9-1-3-#5	$m_3 \oplus m_7 \oplus M_3 \oplus M_3 = M_{13}$
closed fourth inversion	9-1-3-#5-7	$m_7 \oplus M_3 \oplus M_3 \oplus m_3 = M_{13}$
open root position	1-#5-7-9-3	$A_5 \oplus m_3 \oplus m_3 \oplus M_2 = M_{10}$
open first inversion	3-7-9-1-#5	$P_5 \oplus m_3 \oplus m_7 \oplus A_5 = M_{17}$
open second inversion	#5-9-1-3-7	$d_5 \oplus m_7 \oplus M_3 \oplus P_5 = m_{17}$
open third inversion	7-1-3-#5-9	$m_2 \oplus M_3 \oplus M_3 \oplus d_5 = m_{10}$
open fourth inversion	9-3-#5-7-1	$M_2 \oplus M_3 \oplus m_3 \oplus m_2 = m_7$