Coloring Examples in Preparation for Polya–Redfield Theorem

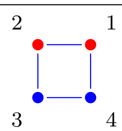
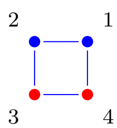
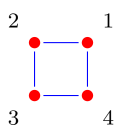
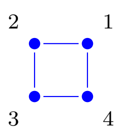
Philip Pennance[1] – Updated: January 20, 2018.

1. **Example 1** Let $V = \{1, 2, 3, 4\}$ and $C = \{C_1, C_2\}$. The permutation $\sigma = (1\,2)(3\,4) \in D_4$ acts on colorings $f \in C^V$ by

$$\sigma * f = f \circ \sigma^{-1}.$$

   (a) The following table lists all colorings fixed by σ:

Colorings fixed by σ

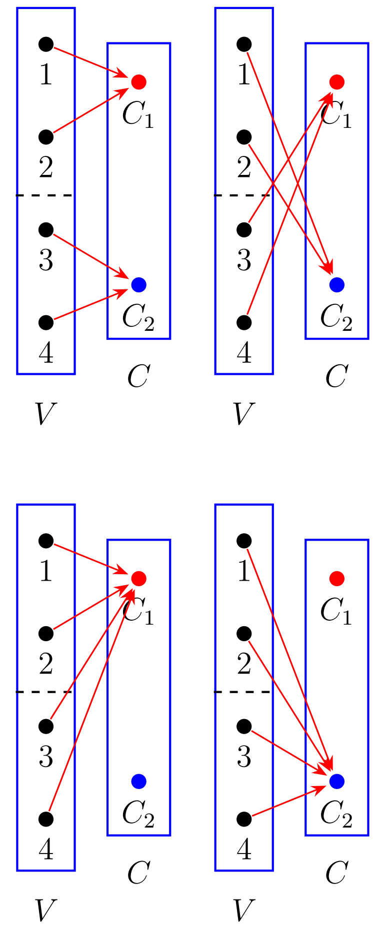
(fixed by σ)	Weight
	$C_1^2C_2^2$
	$C_1^2C_2^2$
	C_1^4
	C_2^4
Weight Inventory	$C_1^4 + 2 \cdot C_1^2C_2^2 + C_2^4$

   (b) Notice that

$$(C_1^2 + C_2^2)^2 = C_1^4 + 2 \cdot C_1^2C_2^2 + C_2^4$$

   is the generating function (weight inventory) of the colorings fixed by σ. This inventory is needed for the application of Burnside's lemma.

   (c) **Remark:** Below are the same four colorings drawn as bipartite graphs:



2. **Example 2** Let $V = \{1, 2, 3, 4, 5, 6, 7\}$ (vertices) and $C = \{C_1, C_2\}$ (colors). The permutation $\sigma = (1\,2)(3\,4)(5\,6\,7)$ acts on colorings $f \in C^V$ by

$$\sigma * f = f \circ \sigma^{-1}.$$

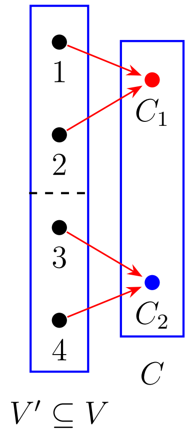
   A weight inventory of the colorings fixed by σ is found as follows:

   (a) A coloring f is fixed by σ if and only if it is constant on each cycle of σ. Thus, for each 2-cycle the contribution is $C_1^2 + C_2^2$ and for each 3-cycle the contribution is $C_1^3 + C_2^3$. The two 2-cycles and the 3-cycle can be handled independently. The contribution from the two 2-cycles $(1\,2)$ and $(3\,4)$ is

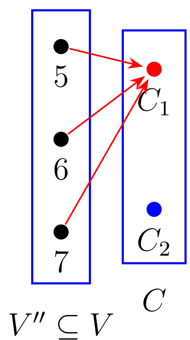
$$(C_1^2 + C_2^2)^2.$$

[1] https://pennance.us

(b) Here is a sample coloring of weight $C_1^2C_2^2$:



(c) Colorings can now be extended to the 3-cycle $(5\,6\,7)$, which contributes $C_1^3 + C_2^3$ to the weight inventory.



(d) The full weight inventory is therefore

$$(C_1^2 + C_2^2)^2(C_1^3 + C_2^3).$$

3. For example, there are two colorings of weight $C_1^5C_2^2$ fixed by σ.

4. More generally, if σ has ν_i cycles of length i, with cycle index

$$P_\sigma = X_1^{\nu_1}X_2^{\nu_2}\cdots X_k^{\nu_k},$$

then the weight inventory of colorings fixed by σ is

$$P_\sigma(Y_1, Y_2, \ldots, Y_k)$$

where

$$Y_k = C_1^k + C_2^k + \cdots + C_m^k.$$

This is the essential idea behind the Redfield–Pólya enumeration theorem.