Polya-Redfield Theorem

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1. Theorem (Redfield).

   Let V be a finite set of vertices and

$$C = \{C_1, C_2 \dots C_m\}$$

a set of colors. Suppose that a permutation group $G \leq S_m$ acts on the set of colorings C^V by

$$\sigma * f = f \circ \sigma^{-1}$$

Define a polynomial P_G (the *cycle index*) as follows. For each permutation $\sigma \in G$ with cycle form $\nu_1 \nu_2 \cdots \nu_k$ let

$$P_\sigma(X_1, X_2, \dots, X_m) = X_1^{\nu_1} X_2^{\nu_2} \cdots X_k^{\nu_k}$$

be the cycle structure of σ. Let

$$\begin{aligned} Y_1 &= \ C_1 + C_2 + \cdots + C_m \\ Y_2 &= (C_1^2 + C_2^2 + \cdots + C_m^2)^2 \\ \vdots &= \qquad\qquad \vdots \\ Y_k &= (C_1 + C_2 + \cdots + C_m)^k \end{aligned}$$

Define

$$P_G = \frac{1}{|G|} \sum_{\sigma \in G} P_\sigma(Y_1, Y_2, \cdots Y_m)$$

Then

(a) P_G is a weight inventory of the colorings $f \in V^C$ fixed by σ which are constant on cycles of σ.

(b) $P_\sigma(m, m, \dots m)$ is the total number $|C_\sigma|$ of colorings fixed by σ.

Proof.

Generalizing the ideas used in these examples shows that

$$\begin{aligned} P_\sigma(Y_1, Y_2, \cdots Y_m) = &(C_1 + C_2 + \cdots + C_m)^{\nu_1} \\ &\times (C_1^2 + C_2^2 + \cdots + C_m^2)^{\nu_2} \\ &\times (C_1^3 + C_2^3 + \cdots + C_m^3)^{\nu_3} \\ &\qquad \vdots \\ &\times (C_1^k + C_2^k + \quad \cdots + C_m^k)^{\nu_k} \end{aligned}$$

is an inventory of the colorings $f \in V^C$ fixed by σ which are constant on cycles of σ. I.e.,

$$P_\sigma(Y_1, Y_2, \dots, Y_k) = \sum_i |X_\sigma^i| W_i$$

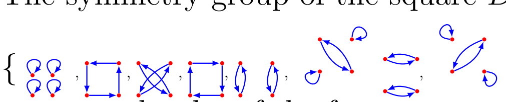
where X_σ^i is the set of colorings of weight W_i which are constant on cycles of σ. It follows, using Burnsides lemma, that each the coefficient r_i is the number of orbits of weight W_i.

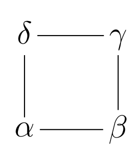
$$\begin{aligned} P_G(Y_1, Y_2, \dots, Y_k) &= \sum_i \left(\frac{\sum_\sigma |X_\sigma^i|}{|G|} \right) W_i \\ &= \sum_i r_i W_i \end{aligned}$$

Finally, part (b) of the theorem is left as a trivial exercise.

2. Example.

   The symmetry group of the square D_4:

{  }
act on molecules of the form



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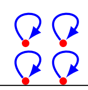
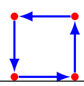
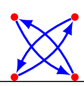
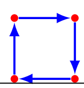
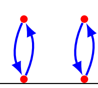
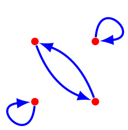
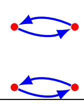
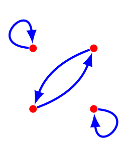
where $\alpha, \beta, \gamma, \delta \in \{C_1, C_2\}$. Let

$$\begin{aligned} Y_1 &= \ C_1 + C_2 \\ Y_2 &= (C_1^2 + C_2^2)^2 \\ Y_3 &= (C_1^3 + C_2^3)^3 \\ Y_3 &= (C_1^4 + C_2^4)^4 \end{aligned}$$

The polynomial P_σ is computed for each σ in the table below:

σ	P_σ	$X_\sigma = P_\sigma(2,2,2,2)$
	X_1^4	16
	X_4^1	2
	X_2^2	4
	X_4^1	2
	X_2^2	4
	$X_1^2 X_2^1$	4
	X_2^2	8
	$X_1^2 X_2^1$	8

For each $\sigma \in D_4$ the number of colorings fixed by σ is calculated using

$$|X_\sigma| = P_\sigma(m,m,m,m)$$

where $m = 2$ is the number of colors. Since

$$\sum_\sigma P_\sigma = X_1^4 + 2X_1^2 X_2^1 + 3X_2^2 + 2X_4^1$$

the cycle index P_G is:

$$P_G(Y_1, Y_2, Y_3, Y_4) = \frac{1}{|G|}\left\{\sum P_\sigma(Y_1, Y_2, Y_3, Y_4)\right\}$$

$$= \frac{1}{8}\left\{(C_1 + C_2)^4 + 2(C_1 + C_2)^2(C_1^2 + C_2^2)^1 \right.$$
$$\left. +3(C_1^2 + C_2^2)^2 + 2(C_1^4 + C_2^4)\right\}$$

$$= C_1^4 + C_1^3 C_2 + 2C_1^2 C_2^2 + C_1 C_2^3 + C_2^4$$

Hence there are 5 weight classes (chemical formulae). The coefficients of P_G count the number of isomers in each weight class. Thus there are two isomers with formula $C_1^2 C_2^2$. These have orbit representatives ,