Residue Theorem

Philip Pennance[1] Version: – May 16, 2003.

1. If a function
$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$
has a Laurent expansion at a point z_0 then a_1 is called the *residue* of f at z_0 and is denoted $\operatorname{Res}(f, z_0)$.

2. Claim.

Let Ω be an open set and $\gamma : [a, b] \to \Omega$ a closed chain homologous to 0 in Ω. If $f \in \operatorname{Hol}(\Omega \setminus \{z_1, z_2, \ldots, z_n\})$, then
$$\int_\gamma f = 2\pi i \sum_{k=1}^{n} W(\gamma, z_k) \operatorname{Res}(f, z_k).$$

Proof.

For each γ_k, $\quad k \in (1, \ldots, N)$ let γ_k be a small enough circle about z_k so that by Cauchy's theorem
$$\int_\gamma f = \sum_{k=1}^{N} W(\gamma_k, z_k) \int_{\gamma_k} f. \qquad (1)$$

Now,
$$\int_{\gamma_k} f = \int_{\gamma_k} \sum_{n \in \mathbb{Z}} a_n (z - z_k)^n \, dz.$$

Since the convergence is uniform, the integral and summation can be exchanged. Hence,
$$\begin{aligned}
\int_{\gamma_k} f &= \sum_{n \neq 1} \int_{\gamma_k} \frac{a_n}{n+1} \left[(z - z_k)^{n+1} \right]' \, dz \\
&\quad + \int_{\gamma_k} \frac{a_{-1}}{z - z_k} \, dz \\
&= \int_{\gamma_k} \frac{a_{-1}}{z - z_k} \, dz \\
&= 2\pi i a_{-1} W(\gamma_k, z_k) \\
&= 2\pi i a_{-1} \\
&= 2\pi i \operatorname{Res}(f, z_k).
\end{aligned}$$

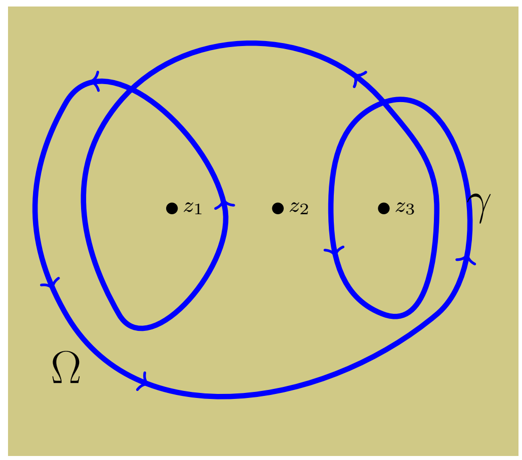
The result follows by substitution this in (1).

Remark.

The second integral does not simplify like the first. (In particular, $\log(z - z_0)$ is not an antiderivative of $\frac{1}{z - z_0}$ on a slit.)

3. Example.

Suppose $f \in \operatorname{Hol}(\mathbb{C} \setminus \{z_1, z_2, z_3\})$ and let γ be the curve drawn below.



Then,
$$\int_\gamma f = 2\pi i \left[2 \operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2) + 2 \operatorname{Res}(f, z_3) \right].$$

[1] https://pennance.us -Source: Lang, Complex Analysis.