Partial Fractions

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1. Definition.

   A point $\alpha \in \mathbb{C}$ is a *singularity* of a complex valued function f if there exists a neighborhood D of α such that f is holomorphic on $D \setminus \alpha$ but not differentiable at α.

2. A singularity α is *removable* if there exists $a \in \mathbb{C}$ such that

   $$z \mapsto \begin{cases} f(\alpha) & \text{if } z \neq \alpha \ , \\ a & \text{if } z = \alpha \end{cases}$$

   is holomorphic in some neighborhood of α.

3. A singularity is a *pole of order* m if there exists a function $g(z)$ holomorphic in a neighborhood of α and $g(\alpha) \neq 0$ such that

   $$f(z) = \frac{g(z)}{(z-\alpha)^m}, \quad z \neq \alpha.$$

   A pole of order 1 is called a *simple pole.*

4. A singularity which is neither removable nor a pole is called essential.

5. A function $f : \mathbb{C} \to \mathbb{C}^*$ is *rational* if there exist polynomials $g, h \in \mathbb{C}[z]$ such that

   $$f(z) = \begin{cases} \dfrac{g(z)}{h(z)} & \text{if } h(\alpha) \neq 0 \ , \\ \infty & \text{if } g(\alpha) \neq 0 \text{ and } h(\alpha) = 0. \end{cases}$$

6. Let $f : \mathbb{C} \to \mathbb{C}^*$ be a rational function.

   $$f(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}.$$

   The extension $\hat{f} : C^* \to C^*$ of a rational function f to C^* is defined by letting

   $$\hat{f}(\infty) = \begin{cases} 0 & \text{if } m > n \ , \\ a_0/b_0 & \text{if } m = n \\ \infty & \text{if } m < n \end{cases} .$$

7. Remark.

   Notice that $\hat{f}(\infty) = f_1(0)$ where f_1 is defined as follows:

   $$f(\tfrac{1}{z}) = \frac{a_0 + a_1 \frac{1}{z} + \cdots + a_n \frac{1}{z^n}}{b_0 + b_1 \frac{1}{z} + \cdots + b_n \frac{1}{z^m}}$$

   $$f_1(z) = \left(\frac{a_0 z^n + a_1^{n-1} + \cdots + a_n}{b_0 z^m + b_1^{m-1} + \cdots + b_m} \right) z^{m-n}$$

8. Definition.

   (a) If $m > n$ we say that ∞ is a *zero of multiplicity* $m - n$.

   (b) If $m < n$ we say that ∞ is a *pole of multiplicity* $n - m$.

9. Claim.

   $$\begin{aligned} |\text{Zeros}(\hat{f})| &= |\text{Poles}(\hat{f})| \\ &= \max(m, n). \end{aligned}$$

   Proof. (By Cases.)

   If $m > n$ then $\hat{f}$ has

   (a) n zeros in $\mathbb{C}$ and $m - n$ at ∞

   (b) m poles in $\mathbb{C}$ and 0 poles at ∞

   and hence $|\text{Zeros}(\hat{f})| = |\text{Poles}(\hat{f})| = m = \max(m, n)$ The other cases are similar.

10. Example.

    Let

    $$f(z) = \frac{z^8 + 6z^5}{z^5}.$$

    Then, up to multiplicity, f has $\max(5, 8) = 8$ poles, 5 in $\mathbb{C}$ and 3 at ∞.

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11. Corollary.

Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a rational function of the form $f(x) = \frac{g(x)}{h(x)}$ where $\deg(g) = n$ and $\deg h = m$. If $\max(m, n) > 0$ then f is surjective.

Proof.

Let $a \in \mathbb{C}^*$. Then

$$f(z) - a = \frac{g(z) - h(z)a}{h(z)}.$$

Without loss it can be assumed that f is in lowest terms.

$$\begin{aligned} |\text{Zeros}|(f - a) &= \max(\deg(g - ha, h) \\ &= \max(\deg g, \deg h) \\ &> 0. \end{aligned}$$

12. Lemma.

A rational function $f : \mathbb{C}^* \to \mathbb{C}^*$ with no poles is constant.

Proof.

If $f : \mathbb{C}^* \to \mathbb{C}^*$ has no poles in $\mathbb{C}$, then it must be a polynomial. If its degree n is greater than zero, then it would have a pole of order n at infinity. Hence f is constant.

13. Claim. (Partial Fraction Expansion)

Let H be a rational function with no pole at infinity ($n < m$). If H has poles $\beta_1, \beta_2, \ldots, \beta_k$, then there exist polynomials $G_1, G_2, \ldots, G_k$ without constant terms such that

$$H(z) = \sum_{i=1}^{k} G_i \left(\frac{1}{z - \beta_i} \right) + C,$$

where C is a constant.

Proof.

Let

$$f_1(w) = \beta_1 + \frac{1}{w}.$$

Then $H \circ f_1(w)$ –as a rational function– has a pole at $w = \infty$. Hence the degree of the numerator is greater than that of the denominator. By division, we can write

$$H \circ f_1(w) = G_1(w) + H_1(w).$$

where G_1 is a polynomial with no constant term and $H_1(w)$ is a rational function which has no pole at ∞. Putting

$$w = f_1^{-1}(z) = \frac{1}{z - \beta_1}$$

gives

$$H(z) - G_1 \left(\frac{1}{z - \beta_1} \right) = H_1 \left(\frac{1}{z - \beta_i} \right)$$

where neither side has a pole at β_1. Iterating this procedure for each of the remaining poles produces

$$H(z) = \sum_{i=1}^{k} G_i \left(\frac{1}{z - \beta_i} \right) + H_k(w).$$

where $H_k(w)$ has no poles in C^*. By the lemma, $H_k(w) = C$ where C is a constant.

14. Example.

Find the partial fraction expansion of

$$H(z) = \frac{z^2}{(z-1)^3}.$$

Solution.

Notice H has a single pole $\beta_1 = 1$ and so only a single iteration of the procedure in claim (13) is required. Specifically, let

$$z = \beta_1 + \frac{1}{w} = 1 + \frac{1}{w}.$$

Then,

$$\begin{aligned} H\left(\beta_1 + \tfrac{1}{w}\right) &= \frac{\left(1 + \frac{1}{w}\right)^2}{\frac{1}{w^3}} \\ &= w + 2w^2 + w^3 \\ &= G_1(w). \end{aligned}$$

Since $w = \frac{1}{z-1}$, it follows that

$$\begin{aligned} H(z) &= G_1\left(\frac{1}{z-1}\right) \\ &= \frac{1}{z-1} + 2\frac{1}{(z-1)^2} + \frac{1}{(z-1)^3}. \end{aligned}$$

15. Example.

Find the partial fraction expansion of

$$H(z) = \frac{z^2}{z^2-1}.$$

Solution.

H has poles $\beta_1 = 1, \beta_2 = -1$.

$$\begin{aligned} H\left(\beta_1 + \frac{1}{w}\right) &= H\left(1 + \frac{1}{w}\right) \\ &= \frac{w^2+2w+1}{2w+1} \\ &= \frac{1}{2}w + H_1(w) \end{aligned}$$

where $H_1(w)$ has no pole at ∞. Similarly,

$$H\left(\beta_2 + \frac{1}{w}\right) = -\frac{1}{2}w + H_2(w)$$

where $H_2(w)$ has no pole at ∞. Use of (13) with

$$G_1(w) = \frac{1}{2}w, \quad G_2(w) = -\frac{1}{2}w,$$

yields

$$\begin{aligned} H(z) &= \sum_{i=1}^{k} G_i\left(\frac{1}{z-\beta_i}\right) + C \\ &= \frac{\frac{1}{2}}{z-1} - \frac{\frac{1}{2}}{z+1} + C. \end{aligned}$$

Putting $z = 0$ shows that $C = 1$.

16. Example.

Find the partial fraction expansion of

$$H(z) = \frac{1}{(z-\beta_1)(z-\beta_2)}.$$

Solution.

$$\begin{aligned} H\left(\beta_1 + \frac{1}{w}\right) &= \frac{1}{(\frac{1}{w})(\frac{1}{w}+\beta_1-\beta_2)} + H_1(w) \\ &= \frac{w^2}{w(\beta_1-\beta_2)+1} + H_1(w) \\ &= \frac{w}{\beta_1-\beta_2} + H_1(w) \end{aligned}$$

where H_1 is a rational fraction with no pole at ∞. Similarly,

$$H\left(\beta_2 + \frac{1}{w}\right) = \frac{w}{\beta_2-\beta_1} + H_2(w),$$

where H_2 is a rational fraction with no pole at ∞. Let

$$G_1(w) = \frac{w}{\beta_1-\beta_2}, \quad G_2(w) = \frac{w}{\beta_1-\beta_2}.$$

Then, by claim (13),

$$\begin{aligned} H(z) &= \sum_{i=1}^{k} G_i\left(\frac{1}{z-\beta_i}\right) + C \\ &= \frac{1}{\beta_1-\beta_2}\frac{1}{z-\beta_1} + \frac{1}{\beta_2-\beta_1}\frac{1}{z-\beta_2} + C. \end{aligned}$$

Putting $z = 1$ yields $C = 1$.

17. Example.

Find the partial fraction expansion of

$$H(z) = \frac{1}{z^4-n^4}.$$

Solution.

H has poles

$$\beta_k = ne^{\frac{k\pi}{2}i}, \quad k \in (0, \ldots, 3).$$

Noting that $b_k^4 = n^4$ we have

$$\begin{aligned} H\left(\beta_k + \frac{1}{w}\right) &= \frac{1}{\left(\beta_k + \frac{1}{w}\right)^4 - n^4} \\ &= \frac{w^4}{1 + 4\beta_k w + 6\beta_k^2 w^2 + 4\beta_k^3 w^3} \\ &= \frac{w}{4\beta_k^3} + H_k(w), \end{aligned}$$

where H_k has no pole at ∞. It follows using (13) that

$$H(z) = \sum_{k=0}^{3} \frac{1}{4\beta_k^3}\left(\frac{1}{z-\beta_k}\right).$$