Laurent Series

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1. Definition.

   An expression,

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

   where $a_n \in \mathbb{C}$, is called a *Laurent series*.

2. A Laurent series converges absolutely (resp. uniformly) on a region $\Omega \subseteq \mathbb{C}$ if the two series

$$f^+(z) = \sum_{n \geq 0} a_n z^n$$

$$f^-(z) = \sum_{n < 0} a_n z^n$$

   converge absolutely (resp. uniformly). In this case,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

   is defined to be the sum $f^+(z) + f^-(z)$.

   Claim.

   Let r, R be numbers with $0 < r < R$ and A, the annulus

$$r \leq |z| \leq R.$$

   If $f \in \mathrm{Hol}(A)$, then f has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

   which converges absolutely and uniformly on

$$-s < |z| < S,$$

   where

$$r < s < S < R.$$

Let C_r and C_R be the circles of radius r and R respectively. Then the coefficients a_n are given by

$$a_n = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{C_R} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta & \text{if } n \geq 0 \ , \\ \\ \dfrac{1}{2\pi i} \displaystyle\int_{C_r} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta & \text{if } n < 0 \ . \end{cases}$$

Proof.

Since f is holomorphic on the closed annulus, then there exists $\epsilon > 0$ such that f is holomorphic on the open annulus Ω

$$r - \epsilon < |z| < R + \epsilon.$$

The closed chain $C_R - C_r$ is homologous to zero in Ω. Hence by Cauchy's formula

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_R - C_r} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta. \end{aligned}$$

The first integral is reduced in the same way as in the proof of Cauchy's formula. Specifically, for $s < |z| < S$

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta} \cdot \frac{1}{(1 - z/\zeta)} \\ &= \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta} \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{\zeta^{n+1}} z^n. \end{aligned}$$

Since $|\frac{z}{\zeta}| < \frac{S}{R} < 1$ the series converges absolutely and uniformly. Hence,

$$\begin{aligned} f^+(z) &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{C_R} \frac{g(\zeta)}{\zeta^{n+1}} d\zeta \right] z^n \\ &= \sum_{n=0}^{\infty} a_n z^n. \end{aligned}$$

[1] https://pennance.us   Source: S. Lang, Complex Analysis, Springer Verlag, 1977

As for the second, let

$$\begin{aligned}\frac{1}{z-\zeta} &= \frac{1}{z} \cdot \frac{1}{(1-\zeta/z)} \\ &= \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{\zeta}{z}\right)^n \\ &= \sum_{n<0} \frac{1}{\zeta^{n+1}} z^n\end{aligned}$$

Since $|\frac{\zeta}{z}| < \frac{r}{s} < 1$ the series converges absolutely and uniformly. Hence,

$$\begin{aligned}f^-(z) &= \sum_{n<0} \left[\frac{1}{2\pi i} \int_{C_r} \frac{g(\zeta)}{\zeta^{n+1}}\, d\zeta\right] z^n \\ &= \sum_{n<0} a_n z^n.\end{aligned}$$

3. Example.

   Find the Laurent series of

$$f(z) = \frac{1}{z(z-1)}$$

   for $0 < |z| < 1$ and for $|z| > 1$.

Solution.

$$f(z) = \frac{1}{z-1} - \frac{1}{z}.$$

If $|z| < 1$,

$$\begin{aligned}\frac{1}{z-1} &= -\frac{1}{1-z} \\ &= -(1 + z + z^2 + \cdots)\end{aligned}$$

and so

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - \cdots .$$

If $|z| > 1$

$$\begin{aligned}\frac{1}{z-1} &= \frac{1}{z}\left(\frac{1}{1-1/z}\right) \\ &= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)\end{aligned}$$

whence

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots .$$