Holomorphicity $\implies$ Analyticity.

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1. Claim.

   Let γ be a path in an open set Ω and g a continuous function on $\operatorname{Im}\gamma$. Define

   $$f : \Omega \setminus \operatorname{Im}\gamma \to \mathbb{C}$$

   by

   $$f(z) = \int_\gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta.$$

   Then, f is analytic on $\Omega \setminus \operatorname{Im}\gamma$ and

   $$f^{(n)}(z) = n! \int_\gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta. \qquad (1)$$

   Proof.

   Let $z_0 \in \Omega \setminus \operatorname{Im}\gamma$ and

   $$r = \min\{d(z_0, \zeta) : \zeta \in \operatorname{Im}\gamma\}.$$

   Then, for $s < r$, and $z \in D(z_0, s)$

   $$\begin{aligned}
   \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0 - (z - z_0)} \\
   &= \frac{1}{\zeta - z_0} \cdot \frac{1}{\left(1 - \dfrac{z - z_0}{\zeta - z_0}\right)} \\
   &= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right) \\
   &= \sum_{n=0}^{\infty} \frac{1}{(\zeta - z_0)^{n+1}} (z - z_0)^n
   \end{aligned}$$

   Since

   $$\left|\frac{z - z_0}{\zeta - z_0}\right| < \frac{q}{s} < 1,$$

   the series converges absolutely and uniformly. Hence,

   $$\begin{aligned}
   f(z) &= \sum_{n=0}^{\infty} \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta (z - z_0)^n \\
   &= \sum_{n=0}^{\infty} a_n (z - z_0)^n,
   \end{aligned}$$

   and so f is analytic. Equation (1) follows by recalling that

   $$a_n = \frac{f^n(z)}{n!}.$$

2. Cauchy formula for first derivative.

   If γ is closed and f is holomorphic then

   $$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

   Proof.

   By the Cauchy formula

   $$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)}\, d\zeta.$$

   The result follows from (1) by putting $g = \dfrac{f(\zeta)}{2\pi i}$ and $n = 1$.

[1] https://pennance.us Source: S. Lang, Complex Analysis.