Cauchy's Theorem - Homological Version

Philip Pennance[1] Version: – April 2003.

1. Introduction.

   Cauchy's theorem is important in the reduction of integrals over complicated paths to integrals over small circles. These can then can evaluated in terms of power series. The theorem reveals that the only obstruction to the vanishing of the integral of a holomorphic function around a closed chain is winding of the chain around some point.

   The following is an outline of E. Artin's proof of Cauchy's Theorem. §(1-14) give some prerequisite material. The source of this material is the book Complex Analysis by Serge Lang. The actual proof begins at §15.

2. Definition.

   Let $\Omega \subseteq \mathbb{C}$ a region. Two curves $\gamma, \eta : [a, b] \to \Omega$ are *close together* if there exists a partition

   $$a = a_0 < a_1 < a_1, \cdots < a_n = b$$

   and disks $D_1, D_2, \ldots, D_n \subseteq \Omega$ such that for all i with $1 \le i \le n$,

   $$\gamma[a_{i-1}, a_i] \subseteq D_i \text{ and } \eta[a_{i-1}, a_i] \subseteq D_i. \tag{1}$$

3. Claim.

   Let $\Omega \subseteq \mathbb{C}$ a region and let γ and η be curves in Ω which are close together and $f \in \mathrm{Hol}(\Omega)$. If γ, η have the same endpoints, or if γ and η are closed, then

   $$\int_\gamma f = \int_\eta f.$$

   Proof.

   Since γ and η are close together, there exist disks $D_i, \quad i = 1, \ldots, n$ satisfying (1). For each i, let F_i be an antiderivative of f on D_i. Since both F_i and F_{i+1} are both antiderivatives on $D_i \cap D_{i+1}$ there exists a constant c such that

   $$F_{i+1}(z) = F_i(z) + c, \quad z \in D_i \cap D_{i+1.}$$

   In particular,

   $$\begin{aligned} F_{i+1}(z_i) &= F_i(z_i) + c \\ F_{i+1}(w_i) &= F_i(w_i) + c. \end{aligned}$$

   Therefore,

   $$F_{i+1}(z_i) - F_{i+1}(w_i) = F_i(z_i) - F_i(w_i).$$

   Now,

   $$\int_\gamma f = \sum_{i=1}^n [F_i(z_i) - F_i(z_{i-1})]$$

   and

   $$\int_\eta f = \sum_{i=1}^n [F_i(w_i) - F_i(w_{i-1})].$$

   Subtracting and noting the presence of telescoping series yields

   $$\begin{aligned} \int_\gamma f - \int_\eta f = &F_n(z_n) - F_n(w_n) \\ &- [F_1(z_0) - F_1(w_0)]. \end{aligned}$$

   It is seen that if either the two curves have equal endpoints, or are both closed, then

   $$\int_\gamma f - \int_\eta f = 0.$$

4. Definition.

   Let $\Omega \subseteq \mathbb{C}$ a region and γ, η closed paths in Ω. We say that γ is *homologous to* η in Ω if

   $$W(\gamma, \alpha) = W(\eta, \alpha), \quad \forall \alpha \in \mathbb{C} \setminus \Omega.$$

[1] https://pennance.us Source: S. Lang, Complex Analysis.

We say that γ is *homologous to zero* in Ω if

$$W(\gamma, \alpha) = 0, \quad \forall \alpha \in \mathbb{C} \setminus \Omega.$$

I.e., γ winds about no point in the complement of Ω.

5. Claim.

Let $\gamma, \eta : [a, b] \to \Omega$ be closed paths. If γ is homotopic to η, then γ is homologous to η.

Proof.

Recall that if γ is homotopic to η then

$$\int_\gamma f = \int_\eta f,$$

for every function $f \in \mathrm{Hol}(\Omega)$. In particular, taking

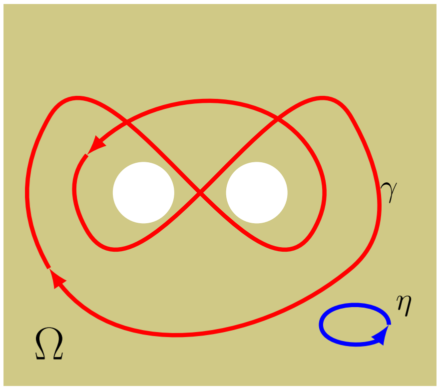
$$f(z) = \frac{1}{2\pi i} \cdot \frac{1}{z - \alpha}, \quad \alpha \in \mathbb{C} \setminus \Omega$$

shows that

$$W(\gamma, \alpha) = W(\eta, \alpha).$$

6. Remark.

The example below shows that the converse of claim (5) is false.



7. Definition.

Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be paths in a region $\Omega \subseteq \mathbb{C}$. A formal sum

$$\sum_{i=1}^{k} n_i \gamma_i, \quad n_i \in \mathbb{Z}$$

is called a *chain*. A chain is called *closed* if each γ_i is closed.

8. Definition.

Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be paths in a region $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ a continuous function. The integral of f along a chain $\gamma = \sum_{i=1}^{k} n_i \gamma_i$ is defined by

$$\int_\gamma f = \sum n_i \int_{\gamma_i} f. \tag{2}$$

9. Definition.

The winding number of a chain

$$\gamma = \sum_{i=1}^{k} n_i \gamma_i$$

about a point $\alpha \in \mathbb{C} \setminus \Omega$ is defined by

$$W(\gamma, \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - \alpha}\, dz$$

the integral being evaluated using (2).

10. Definition.

Two closed chains γ and η are *homologous* in a region $\Omega \subseteq \mathbb{C}$ if

$$W(\gamma, \alpha) = W(\eta, \alpha), \quad \forall \alpha \in \mathbb{C} \setminus \Omega.$$

11. Definition.

Let $\Omega \subseteq \mathbb{C}$ be a region and

$$\gamma : [a, b] \to \Omega$$

a path. Let

$$a = a_0 < a_1 < a_2 < \ldots a_n = b$$

be a partition of $[a, b]$. Let

$$\gamma_i : [a_{i-1}, a_i] \to \Omega$$

be the restriction of γ to $[a_{i-1}, a_i]$. The chain

$$\eta = \gamma_1 + \gamma_2 + \cdots \gamma_n$$

is called a subdivision of γ.

12. Definition.

    Let γ be a chain. A chain η is a *subdivision* of γ written $\eta = \operatorname{top}\gamma$ if η is obtained from γ by subdividing paths in γ.

13. Corollary.

    If $\eta = \operatorname{top}\gamma$, then

    $$\int_\gamma f = \int_\eta f.$$

14. Lemma.

    Let γ be a path in an open set Ω. Then, there exists a rectangular path η with the same endpoints such that γ and η are close together.

    Proof.

    Since $\operatorname{Im}\gamma$ is compact, there exists a covering by a finite number of open disks $D_1, D_2, \ldots D_n$ and a partition

    $$a = a_0 < a_1 < a_2 < \cdots < a_n = b$$

    such that

    $$\gamma_i[a_i - 1, a_i] \subseteq D_i.$$

    The restriction of γ to each disk can clearly be replaced by a rectangular curve within the disk.

15. Cauchy's Theorem.

    Let $\Omega \subseteq \mathbb{C}$ be open and γ a closed chain in Ω. If γ is homologous to zero then

    $$\int_\gamma f = 0$$

    for any function f which is holomorphic in Ω.

    Proof. (E. Artin).

    By (14) there exists a rectangular closed path η close to γ. Therefore,

    $$\int_\gamma f = \int_\eta f$$

    for every f holomorphic on Ω. In particular, if $\alpha \in \mathbb{C} \setminus \Omega$ then

    $$\int_\eta \frac{1}{z-\alpha}\,dz = \int_\gamma \frac{1}{z-\alpha}\,dz$$

    and so, if γ is homologous to zero, then so is η. Hence, we may assume from the outset that γ is a rectangular path. Indeed, the proof is finished if we can show, in this case, that there exist rectangles $R_1, R_2, \ldots, R_n \subseteq \Omega$ and non zero integers $m_1, m_2, \ldots, m_n$ such that

    $$\sum_{i=1}^n m_i \partial R_i = \operatorname{top}\gamma$$

    for, by the theorem of Cauchy and Gourset, since f is holomorphic on each R_i,

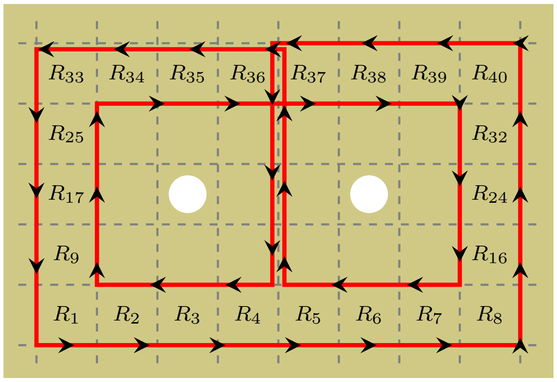
    $$\begin{aligned}\int_\gamma f &= \int_{\sum m_i \partial R_i} f\\ &= \sum m_i \int_{\partial R_i} f\\ &= 0.\end{aligned}$$

    The desired set of rectangles $\mathcal{R}$, some of them unbounded, is obtained as follows.

    (a) Draw horizontal and vertical lines through each edge of the rectangular curve γ.

    (b) For each rectangle R_i formed in step (a), pick $\alpha_i \in \mathring{R}_i$ (the interior) and let $m_i = W(\gamma, \alpha_i)$.

    (c) Let $R_1, R_2, \ldots R_N$ *after a possible renumbering,* be the rectangles with $m_i \neq 0$.



Rectangles R_i with $m_i \neq 0$
(before relabeling)

Subclaim.

Ω contains the rectangle $R_1, R_2, \ldots, R_N$ together with their boundaries.

Proof.

Let $\alpha \in \mathbb{C} \setminus \Omega$. Since γ is homolologous to zero in Ω

$$W(\gamma, \alpha) = 0, \quad \forall \alpha \in \mathbb{C} \setminus \Omega$$

Suppose (by contradiction) that $\alpha \in R_i$ for some i.

(a) If $\alpha \in \mathring{R}_i$ then

$$\begin{aligned} W(\gamma, \alpha) &= W(\gamma, \alpha_i) \\ &= m_i \\ &\neq 0, \end{aligned}$$

which is a contradiction.

(b) If $\alpha \in \partial R_i \cap \operatorname{Im} \gamma$, then $\alpha \in \Omega$ which is a contradiction.

(c) If $\alpha \in \partial R_i \setminus \operatorname{Im} \gamma$, then

$$W(\gamma, \alpha) = W(\gamma, \alpha_i) \neq 0,$$

yet another contradiction.

Since α belongs to none of the R_i the subclaim is proven.

It remains to show.

$$\sum m_i \partial R_i = \operatorname{top} \gamma.$$

Subclaim.

Let η be the subdivision of γ with path set determined by the trace of the path γ in the edge set of the rectangles and let

$$C = \eta - \sum m_i \partial R_i.$$

Then,

$$W(C, \alpha) = 0, \quad \forall \alpha \text{ not on } C. \tag{3}$$

Proof.

Since η is closed, C is a closed chain and

$$W(C, \alpha) = W(\eta, \alpha) - m_i W(\partial R_i, \alpha).$$

(a) If α is in one of the unbounded rectangles, then the constancy of W in such a region, together with the fact that $\frac{1}{z-\alpha} \to 0$ as $z \to \infty$ means that the constant must be zero.

(b) If $\alpha \in \mathring{R}_k, \quad 1 \leq k \leq N$, then

$$W(\eta, \alpha) = W(\eta, \alpha_k) = m_k$$

and

$$W(\partial R_i, \alpha) = \begin{cases} 1 & \text{if } i = k \ , \\ 0 & \text{if } i \neq k. \end{cases}$$

It follows that

$$W(C, \alpha) = m_k - m_k = 0.$$

(c) If $\alpha \in R \in \mathcal{R} \setminus \{R_1, R_2, \ldots R_N\}$, then $W(\eta, \alpha) = W(\gamma, \alpha) = 0$ and $W(\partial R_i, \alpha) = 0$. Again, it is seen that $W(C, \alpha) = 0$.

(d) If α lies on the boundary of any rectangle but not on C then $W(C, \alpha) = 0$ by continuity of W in the regions determined by C.

Thus in all cases (3) holds.

Subclaim.

$$\eta = \sum m_i \partial R_i$$

I.e., $C = 0$.

Proof.

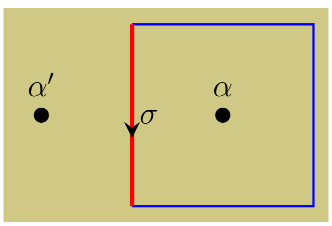
Suppose to the contrary that there exists a segment $\sigma \in C$. Then,

$$C = m\sigma + C^*.$$

for some integer $m \neq 0$ and chain C^* not containing σ. There are two cases:

Case 1.

σ is side of an finite rectangle R .



Without loss, we can orient σ to coincide with the orientation of ∂R. Let $\alpha \in R$, and $\alpha' \notin R$ be points close to and on opposite sides of σ. Since σ does not belong to the closed chain $C - m\partial R$, it follows (from the constancy of W on the regions determined by $C - m\partial R$) that

$$\begin{aligned} W(C - m\partial R, \alpha) &= W(C - m\partial R, \alpha') \\ &= W(C, \alpha') \end{aligned}$$

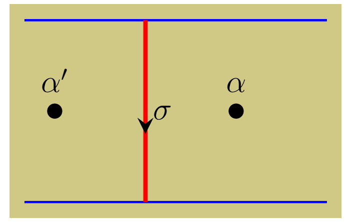
Therefore,

$$W(C, \alpha) - mW(\partial R, \alpha) = W(C, \alpha')$$

Since $W(\partial R, \alpha) = 1$ and $W(C, \alpha') = W(C, \alpha) = 0$, it follows that $m = 0$, which is a contradiction.

Case 2.

σ separates two infinite rectangles



Since $C = m\sigma + C^*$,

$$W(C, \alpha) = \frac{m}{2\pi i}\left[\int_\sigma \frac{1}{z - \alpha}\,dz + \int_{C^*} \frac{1}{z - \alpha}\,dz\right]$$

and

$$W(C, \alpha') = \frac{m}{2\pi i}\int_\sigma \frac{1}{z - \alpha'} + \frac{m}{2\pi i}\int_{C^*} \frac{1}{z - \alpha'}$$

Hence by (3)

$$\int_\sigma \frac{1}{z - \alpha} - \frac{1}{z - \alpha'}\,dz = \int_{C^*} \frac{1}{z - \alpha} - \frac{1}{z - \alpha'}\,dz$$

Since $\sigma \notin C^*$ the right hand side is a constant on the segment $[\alpha, \alpha']$. Letting $\alpha \to \alpha'$ shows that the constant is zero. Direct integration over a segment shows that the integral on the left is non zero. It can only be that $m = 0$ which is impossible. This completes the proof that $\eta = \sum m_i R_i$. Since $\eta = \text{top}\,\gamma$ Cauchy's theorem is proven.

16. Corollary.

Let $\Omega \subseteq \mathbb{C}$ be a region and γ, η be closed chains. If γ homologous to η in Ω. If $f \in \text{Hol}(\Omega)$ then

$$\int_\gamma f = \int_\eta f.$$

Proof.

Since $\gamma - \eta$ is a closed chain, Cauchy's theorem implies that

$$\int_{\gamma - \eta} f = 0.$$

17. Remarks.

Let $\Omega \subseteq \mathbb{C}$ be a region and γ a curve homologous to 0 in Ω. In order to apply Cauchy's theorem on the region Ω a function f must be holomorphic on Ω. If f is holomorphic only on a subset

$$\Omega^* = \Omega \setminus \{z_1, z_2, \ldots, z_n\}.$$

and say $W(\gamma, z_i) \neq 0$ for some i, the theorem cannot be applied to the curve γ which is not homologous to zero on Ω^*. The following corollary shows that in this situation, there exists a chain homologous to 0 on Ω^* which permits application of the theorem.

18. Corollary.

Let $\Omega \subseteq \mathbb{C}$ be a region and γ a closed chain homologous to 0 in Ω. If f is holomorphic only on a subset

$$\Omega^* = \Omega \setminus \{z_1, z_2, \ldots, z_n\}.$$

Then, there exists a closed chain $\sum_{i=1}^n m_i C_i$ on Ω^* such that

$$C = \gamma - \sum m_i C_i$$

is homologous to 0 in Ω^* and, in particular,

$$\int_\gamma f = \sum_{i=1}^n m_i \int_{C_i} f. \tag{4}$$

Proof.

For each i, C_i be a counter clockwise oriented circle surround z_i and small enough so as not to intersect the curve γ. The C_i can clearly be chosen to be disjoint. Let $m_i = W(\gamma, z_i)$ define a closed chain,

$$C = \gamma - \sum m_i C_i$$

We claim that C is homologous to 0 in Ω^*. Let α any point in $\mathbb{C} \setminus \Omega^*$. If $\alpha \notin \{z_1, z_2, \ldots, z_n\}$, then $W(C_i, z_i) = 0$ $i \in (1, \ldots, n)$. Also, in this case $\alpha \in \mathbb{C} \setminus \Omega$ and so, by hypothesos $W(\gamma, \alpha) = 0$. Hence,

$$\begin{aligned} W(C, \alpha) &= W(\gamma, \alpha) - \sum m_i W(C_i, \alpha) \\ &= 0. \end{aligned}$$

On the other hand, if $\alpha = z_k$ for some k, since $\alpha \in \Omega$, $W(\gamma, \alpha) = m_k$ and

$$W(\partial C_i, \alpha) = \begin{cases} 1 & \text{if } i = k \ , \\ 0 & \text{if } i \neq k \end{cases}$$

Once again, $W(C, \alpha) = 0$. Thus C is homologous to zero in Ω and equation (4) follows.

19. Example

Let γ be the curve drawn below. Let $\Omega^* = \mathbb{C} \setminus \{z_1, z_2, z_3\}$. Let C_1, C_2, C_3 be small circles centered at z_1, z_2, z_3 respectively and oriented counterclockwise. Since

$$C = \gamma + 2C_1 + C_2 + C_3$$

is homologous to 0 and so if $f \in \operatorname{Hol} \Omega^*$, then

$$\int_\gamma f = -2 \int_{C_1} f - \int_{C_2} f - 2 \int_{C_3} f$$