Cauchy Formulae: Examples

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1. Cauchy's Formula (Reminder).

$$\int_\zeta \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z).$$

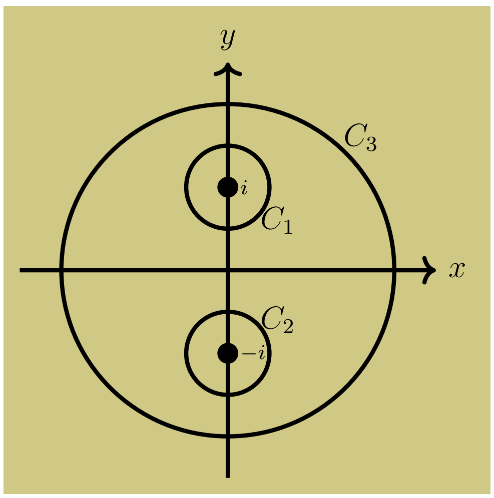
2. Example.

   Evaluate
$$\int_C \frac{z}{z^2+1}\, dz$$
   for each of the following circles.

   (a) $C_1 : |z - i| = \frac{1}{2}$.

   (b) $C_2 : |z + i| = \frac{1}{2}$.

   (c) $C_3 : |z| = 2$.



   Solutions.

   (a) Let
$$f(z) = \frac{z}{z+i}.$$
   Then f is holonomic inside C_1.

   Hence,
$$\int_{C_1} \frac{z}{z^2+1}\, dz = \int_{C_1} \frac{f(z)}{z-i}\, dz = 2\pi i f(i) = \pi i.$$

   (b) Let
$$f(z) = \frac{z}{z-i}.$$
   Then, f is holonomic inside C_2.

   Hence,
$$\int_{C_2} \frac{z}{z^2+1}\, dz = \int_{C_1} \frac{f(z)}{z+i}\, dz = 2\pi i f(-i) = \pi i.$$

   (c)
$$\frac{z}{z^2+1} = \frac{\frac{1}{2}}{z+i} + \frac{\frac{1}{2}}{z-i}.$$
   Taking $f(z) = 1$ in Cauchy's formula yields,
$$\int_{C_3} \frac{z}{z^2+1}\, dz = \tfrac{1}{2} \cdot 2\pi i \cdot f(-i) + \tfrac{1}{2} \cdot 2\pi i . f(i) = 2\pi i.$$

3. Cauchy's Derivative Formula .

4. Example.

   Find $\displaystyle\int_{|z=7|} \frac{e^{2z}}{(z+1)^4}\, dz.$

   Solution.

   The integrand has a pole of order 4 at $z = -1$ which lies inside the circle $|z| = 7$. Using the Cauchy derivative formula
$$\int_\gamma \frac{f(z)}{(z-i)^{n+1}} = \frac{2\pi i f^{(n)}(a)}{n!}$$
   with $f(z) = e^{2z}$, $z = -1$ and $n = 3$ yields
$$\int_{|z|=7} \frac{f(z)}{(z-i)^4}\, dz = \frac{2\pi i f'''(-1)}{3!} = \frac{2\pi i \cdot 8e^{-2}}{3!} = \frac{8}{3}\pi i e^{-2}.$$

[1] https://pennance.us