Uniform Convergence

Philip Pennance[1] Version: – April 2003.

1. Definition.

   A sequence $(f_n)_{n\in\mathbb{N}}$ of complex valued functions on a set $\Omega \subseteq \mathbb{C}$ is *pointwise convergent* with limit $f : \Omega \to \mathbb{C}$ on Ω if for all $x \in \Omega$ and for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$

   $$|f_n(x) - f(x)| < \epsilon.$$

2. Remark.

   The pointwise limit of a sequence of continuous functions is not necessarily continuous. Hence a stronger mode of convergence is required.

3. Definition.

   A sequence $(f_n)_{n\in\mathbb{N}}$ of complex valued functions on a set $\Omega \subseteq \mathbb{C}$ is *uniformly convergent* on Ω with limit $f : \Omega \to \mathbb{C}$ if for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$ and $x \in \Omega$.

   $$|f_n(x) - f(x)| < \epsilon.$$

4. Remark.

   Uniform convergence is just pointwise convergence with respect to the supremum metric, defined by

   $$d(f, g) = \|f - g\| = \sup_{x\in\Omega} |f(x) - g(x)|.$$

   Specifically, if $d_n = \|f_n - f\|$, then f_n converges to f uniformly if $d_n \to 0$ as $n \to \infty$.

5. Notation.

   If (f_n) converges uniformly to f we write $f_n \rightrightarrows f$.

6. Claim.

   Suppose for every $\epsilon > 0$, there exists a natural number N such that

   $$m, n \geq N \implies ||f_m - f_n|| < \epsilon.$$

   Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly.

   I.e., a sequence of functions which is Cauchy in the supremum norm converges uniformly.

   Proof.

   By hypothesis, there exists a natural number N such that for all $z \in \Omega$. and $n > N$,

   $$\begin{aligned} |f_n(z) - f_m(z)| &< \|f_n - f_m\| \\ &< \frac{\epsilon}{2}. \end{aligned}$$

   Hence $(f_n(z)$ is a Cauchy sequence in $\mathbb{C}$ and so (by completeness) converges to a function $f(z)$. It follows that there exists an integer $M > N$ such that for all integers $m > M$

   $$|f_m(z) - f(z)| < \frac{\epsilon}{2}.$$

   Then, for all natural numbers $n > N$

   $$\begin{aligned} |f_n(z) - f(z)| &\leq |f(z) - f_m(z)| \\ &\quad + |f_m(z) - f(z)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

   Since N is independent of z, the convergence is uniform. (Note that M will generally depend on z.)

7. Definition.

   A series of functions $\sum f_i$ is said to *converge uniformly* if the sequence of partial sums converges uniformly.

[1] https://pennance.us   Reference: Protter and Morrey, Modern Mathematical Analysis.

8. Claim. (M-test or Comparison test).

   Let $(f_n) : \Omega \to \mathbb{C}$ be a sequence of functions. Suppose that there exist a sequence $(c_n)_{n\in\mathbb{N}}$ of non-negative real numbers such that

   $$\|f_n\| \leq c_n.$$

   If $\sum c_n$ converges, then $\sum f_n$ converges uniformly and absolutely on Ω.

   Proof.

   We show that the sequence (s_n) of partial sums is Cauchy.

   $$\begin{aligned}\|s_m - s_n\| &= \|\sum_{k=m+1}^{n} f_k\| \\ &\leq \sum_{k=m+1}^{n} \|f_k\| \\ &\leq \sum_{k=m+1}^{n} c_k.\end{aligned}$$

   Since (c_n) converges it must be Cauchy. It follows that (s_n) is also Cauchy. By (6) (s_n) converges uniformly. Moreover, since

   $$\sum |f_n(z)| \leq \sum c_n$$

   the convergence is absolute.

9. Claim.

   The series
   $$\sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n$$
   converges uniformly on $|z| < |a|$.

   Proof.

   Let $z \in \mathbb{C}$ with $|z| < |a|$ and let r such that $|z| \leq r < |a|$. Then on the closed disk $|z| \leq r$

   $$\begin{aligned}\|f_n\| &= \max\left\{\left|\left(\frac{z}{a}\right)^n\right| : |z| \leq r\right\} \\ &\leq \left(\frac{r}{|a|}\right)^n < 1\end{aligned}$$

   Taking $c_n = \left(\frac{r}{|a|}\right)^n$ in the M-test. and noting that $\sum c_n$ converges, it follows that $\sum f_n$ converges uniformly and absolutely on $|z| \leq r$ for any $r < |a|$ and hence on the open disk $D(0, a)$.

10. Example.

    Let $\Omega \subseteq \mathbb{C}$ be bounded. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is uniformly convergent on Ω.

    Solution.

    Since Ω is bounded, there exists a closed disk D of radius r center the origin, containing Ω. On D, $\left|\frac{z^n}{n!}\right| \leq \left|\frac{r^n}{n!}\right|$. Let $c_n = \left|\frac{r^n}{n!}\right|$. The ratio test shows that $\displaystyle\sum_{n=0}^{\infty} c_n$ converges. By the M-test the original series is convergent on D and since $\Omega \subseteq D$ on Ω.

11. Claim.

    Suppose that the power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges for some $\alpha \in \mathbb{C}$ with $|\alpha| = r$ Then, the series converges absolutely for $|z| < r$ and uniformly on $|z| \leq s$ for any s with $s < |\alpha|$.

    Proof.

    Let $f_n(z) = a_n z^n$. Then, for (all) z with $|z| \leq r$

    $$\begin{aligned}|f_n(x)| &= |a_n z^n| \\ &\leq |a_n r^n| \\ &< |a_n| s^n.\end{aligned}$$

    Hence,

    $$\begin{aligned}\|f_n\| &= \sup\{f_n(z) : |z| \leq r\} \\ &\leq |a_n| s^n.\end{aligned}$$

    By the M-test with $c_n = |a_n| s^n$ original series converges absolutely for $|z| < r$ and uniformly on $|z| < s$.

12. Claim. (Uniform Limit Theorem)

Let (f_n) be a sequence of continuous functions. If (f_n) converges uniformly to f then f is continuous.

Proof.

Let $\epsilon > 0$ and $a \in \Omega$. We seek $\delta > 0$ such that

$$0 < |z-a| < \delta \implies |f(z) - f(a)| < \epsilon.$$

Since $f_n \rightrightarrows f$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $z \in \Omega$

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}.$$

In particular,

$$|f_N(a) - f(a)| < \frac{\epsilon}{3}.$$

and

$$|f_N(z) - f(z)| < \frac{\epsilon}{3}.$$

Since f_N is continuous at a, there exists $\delta > 0$ such that

$$0 < |z-a| < \delta \implies |f_N(z) - f_N(a)| < \frac{\epsilon}{3}.$$

Thus if $0 < |z-a| < \delta$ then

$$\begin{aligned} |f(z) - f(a)| &\leq |f(z) - f_N(a)| \\ &\quad + |f_N(z) - f_N(a)| \\ &\quad + |f_N(a) - f_N(a)| \\ &< \epsilon. \end{aligned}$$

13. Definition.

Let $\Omega \subseteq \mathbb{C}$ be a region and $f : \Omega \to \mathbb{C}$. We say that f *admits a power series represenation at* $z_0 \in \mathbb{C}$ if there exists a power series $\sum a_n(z-z_0)^n$ convergent to $f(z)$ on some disk $D(z_0, r) \subseteq \Omega$.

14. Definition.

Let $\Omega \subseteq \mathbb{C}$ be a region and $f : \Omega \to \mathbb{C}$. f is *analytic on* Ω if for all $z_0 \in \Omega$, f admits a power series representation at z_0.

15. Claim.

Let $\Omega \subseteq \mathbb{C}$ be a region and $(f_n)_{n\in\mathbb{N}}$ a sequence of complex valued functions on Ω. Suppose that

(a) For all n, f_n is continuous.

(b) $f_n \to f$ uniformly on Ω.

Then for any smooth curve γ in Ω

$$\int_\gamma f_n \to \int_\gamma f.$$

I.e.,

$$\lim \int_\gamma f_n = \int_\gamma \lim f_n.$$

Proof.

The hypothese guarantee that f is continuous and so $\int_\gamma f$ exists.

$$\begin{aligned} \left| \int_\gamma f_n - \int_\gamma f \right| &= \left| \int_\gamma (f_n - f) \right| \\ &\leq \int_\gamma |f_n - f| \\ &\leq \int_\gamma \|f_n - f\| \\ &\leq \|f_n - f\| \ell(\gamma). \end{aligned}$$

Since $f_n \to f$ uniformly, $\|f_n - f\| \to 0$ and so

$$\int_\gamma f_n \to \int_\gamma f.$$

16. Corollary. (A power series can be integrated term by term.)

Suppose that

$$f(z) = \sum a_n z^n, \quad |z| < R.$$

(meaning that the sequence of partial sums converges uniformly to f.) Then,

$$\int_\gamma \sum_{n=0}^{\infty} a_n z^n \, dz = \sum_{n=0}^{\infty} \frac{a_n z^n}{n+1} a_n z^n. \quad (1)$$

Proof.

Let $f_n = \sum_{k=0}^{n} a_k z^k$.

$$
\begin{aligned}
& f_n \rightrightarrows f \text{ on } |z| < \mathbb{R} \\
\implies & \int_\gamma f_n \to \int_\gamma f \\
\implies & \int_\gamma \sum_{k=0}^{n} a_k z^k \, dz \to \int_\gamma \sum_{n=0}^{\infty} z^n \, dz \\
\implies & \sum_{k=0}^{n} a_k \int_\gamma z^k \, dz \to \int_\gamma \sum_{n=0}^{\infty} a_n z^n \, dz.
\end{aligned}
$$

Since z^k is analytic on $|z| < R$ the integral $\int_\gamma z^k dz$ is independent of path. Hence,

$$
\begin{aligned}
\int_\gamma z^k \, dz &= \int_{[0,z]} z^k \, dz \\
&= \frac{z^{k+1}}{k+1}.
\end{aligned}
$$

and (1) follows.