Limit superior and limit inferior

© Prof. Philip Pennance

1. Definition. The limit superior of a sequence \((a_n)\) is the extended real number defined by

\[
\limsup a_n = \lim_{n \to \infty} \left( \sup_{k \geq n} a_k \right)
\]

Similarly, the limit inferior of \((a_n)\) is defined by

\[
\liminf a_n = \lim_{n \to \infty} \left( \inf_{k \geq n} a_k \right)
\]

2. Existence: Let

\[ T_i = \{ a_k : k \geq i \} \]

be the i’th tail. Notice that

\[ T_1 \supseteq T_2 \supseteq \cdots \]

Hence \( \sup T_m \) is always non-increasing and \( \inf T_m \) non-decreasing. It follows that

\[
\limsup a_n = \lim_{n \to \infty} \sup T_n = \inf_n (\sup T_n)
\]

and

\[
\liminf a_n = \lim_{n \to \infty} (\inf T_n) = \sup_n (\inf T_n)
\]

always exist as extended real numbers and that

\[
\liminf a_n \leq \limsup a_n \quad (1)
\]

3. Remark: For all \( m, n \in \mathbb{N} \):

\[
\inf T_n \leq \inf T_{\max\{m, n\}} \leq \sup T_{\max\{m, n\}} \leq \sup T_m
\]

which also implies equation (1).

4. Notation.

\( \limsup a_n \) and \( \liminf a_n \) are often denoted \( \lim \sup \) and \( \lim \inf \) respectively.

5. Example.

Let

\[
a_n = \begin{cases} 
\frac{1}{n} & \text{if } n \text{ is odd} \\
-1 & \text{if } n \text{ is even}
\end{cases}
\]

Then \( \sup T_n = 1/n \) and \( \inf T_n = -1 \). Hence

\[
\limsup a_n = 0 \quad \liminf a_n = -1.
\]

6. Claim. Let \((a_n)\) be a bounded sequence and \(A', A'' \in \mathbb{R}\) such that

\[
A' < \limsup a_n < A''
\]

then

(a) \( a_n > A'' \) for (at most) finitely many \( n \).

(b) \( a_n > A' \) for infinitely many \( n \).

Proof. Exercise

1 http://pennance.us
7. Alternative characterization of the limit superior.
   Let \((a_n)\) be a sequence. Then \(\limsup a_n\) is the unique extended real number number \(L\) with the property that for all \(A', A'' \in \mathbb{R}\) with \(A' < L < A''\), the following hold:
   (a) \(a_n > A''\) for (at most) finitely many \(n\).
   (b) \(a_n > A'\) for infinitely many \(n\).

Proof. The previous claim shows that \(L = \limsup a_n\) satisfies the property. If the property is also true for \(L\neq L^*\) it is an easy exercise to see that an obvious contradiction arises:

8. Corollary. The limit superior of \((a_n)\) is the smallest real number \(L\) such that, for any positive real number \(\epsilon\) there exists a natural number \(N\) such that \(a_n < L + \epsilon\) for all \(n > N\)

Proof. Exercise.

9. Dual property.
   \(\liminf a_n\) is the unique extended real number \(L\) such that for all \(A', A'' \in \mathbb{R}\) with \(A' < L < A''\), the following hold:
   (a) \(a_n < A'\) for (at most) finitely many \(n\).
   (b) \(a_n < A''\) for infinitely many \(n\).

10. Corollary. The limit inferior of \((a_n)\) is the largest extended real number \(L\) such that, for any positive real number \(\epsilon\) there exists a natural number \(N\) such that \(a_n > L - \epsilon\) for all \(n > N\).

11. Definition. Let \(\alpha = (a_n)\) be a bounded sequence. An element \(b \in \mathbb{R}\) is an eventual upper bound of \(\alpha\) if \(b < a_n\) for at most finitely many \(n\).

12. Let \(B_{\alpha}\) be the set of all eventual upper bounds of \(\alpha\). Then
   (a) \(B_{\alpha}\) contains all upper bounds of \(\alpha\)
   \[UB(\alpha) \subseteq B_{\alpha}\]
   (b) \(B_{\alpha}\) is absorbing from the right, i.e.,
   \[x > b \in B_{\alpha} \Rightarrow x \in B_{\alpha}\]
   (c) \(\inf B_{\alpha} = \limsup a_n\)

Proof of (c).
   Let \(B^* = \inf B_{\alpha}\) and suppose that \(A', A'' \in \mathbb{R}\) satisfy
   \[A' < B^* < A''\]
   Then there exists \(b \in B_{\alpha}\) such that \(B^* \leq b < A''\). It follows by right absorption that \(A'' \in B_{\alpha}\) and so \(A'' < a_n\) for at most finitely many \(n\). On the other hand
   \[A' < \inf B_{\alpha}\]
   \[\Rightarrow A' \notin B_{\alpha}\]
   \[\Rightarrow A' < a_n\] for infinitely many \(n\)
   Thus \(B^*\) satisfies the characterization of the limit superior in (7). Thus the limit superior of a sequence is the infimum of the set of eventual upper bounds.

13. Warning. Some sources describe the limit superior as an “eventual” upper bound. This terminology is misleading. For example, \(\limsup \frac{1}{n} = 0\) yet 0 is a lower bound for the sequence.

14. Exercise. Use the alternative characterizations of limit superior and limit inferior to give another proof of (1).

15. If \(\lim a_n\) exists then
   \[\liminf a_n = \lim a_n = \limsup a_n\]

Proof.
   Notice the \(a_n\) satisfies the alternative characterizations of the limits superior and inferior given in (7) and (8).
16. If \( \liminf a_n = \limsup a_n \) then the sequence \((a_n)\) converges to their common value.

Proof. From (7), (8) above, it follows that any open interval containing the common value contains all but a finite number of terms of the sequence.

17. Claim.

\[
\limsup(a_n + b_n) \leq \limsup(a_n) + \limsup(b_n)
\]

(2)

Proof: Recall that for any two sequences \((a_n)\) and \((b_n)\)

\[
\sup_{n} \{a_n + b_n\} \leq \sup_{n} a_n + \sup_{n} b_n.
\]

In particular, applying this to the \(n\)-th tails of the two sequences

\[
\sup_{k \geq n} \{a_k + b_k\} \leq \sup_{k \geq n} \alpha + \sup_{k \geq n} \beta.
\]

Since this holds for all \(n\) the limit can be taken on both sides:

\[
\limsup(a_n + b_n) = \lim_{n \to \infty} \left( \sup_{k \geq n} \{a_k + b_k\} \right)
\]

\[
\leq \lim_{n \to \infty} \left( \sup_{k \geq n} (a_k) + \sup_{k \geq n} (b_k) \right)
\]

\[
= \limsup(a_n) + \limsup(b_n)
\]

Alternative Proof. Let \(\alpha = \limsup a_n\) and \(\beta = \limsup b_n\). Then \(\exists N, N'\) such that

\[
n > N \Rightarrow a_n < \alpha + \epsilon/2
\]

\[
n > N' \Rightarrow b_n < \beta + \epsilon/2
\]

and so

\[
n > N + N' \Rightarrow a_n + b_n < \alpha + \beta + \epsilon
\]

\(a_n + b_n\) for finitely many \(n\)

18. Mean Property

Let \((a_n)\) be a bounded sequence. Define \((x_n)\) by

\[
x_n = \frac{a_1 + a_2 + \cdots + a_n}{n}
\]

Then

\[
\limsup x_m \leq \limsup a_n
\]

with an obvious dual statement for the limit inferior.

Proof. Let \(\epsilon > 0\). Then, there exists \(N\) such that

\[
n > N \Rightarrow x_n < \limsup a_n + \epsilon
\]

Let \(U = \limsup a_n\) and \(n > N\). Then

\[
x_n < \frac{a_1 + a_2 + \cdots + a_N + (n - N)[U + \epsilon]}{n}
\]

(3)

Notice that for all \(n\) sufficiently large,

\[
x_n < U + \epsilon
\]

This implies that \(\limsup x_n \leq U\).

(Remark: This also follows by applying property (2) to equation (3).


A bounded convergent sequence has a convergent subsequence.

Proof. Let \((a_n)\) be bounded. The idea is to construct a subsequence converging to \(L = \limsup a_n\).

\(\exists n_1\) such that \(a_{n_1} > L - 1\). Proceeding by recursion, suppose that there exist natural numbers

\[
n_1 < n_2 < \cdots < n_k
\]
such that

\[ a_{n_j} > L - 1/j, \quad 1 \leq j \leq k \]

It is not assumed that the \( a_{n_j} \) are increasing. However, since \( a_{n_k} < L - 1/j \) for finitely many \( n_k \)

\[ L - \frac{1}{j} \quad L \]

it follows that \( L \leq \lim \inf a_{n_k} \) and so

\[ L \leq \lim \inf a_{n_k} \leq \lim \sup a_{n_k} \leq L \]

proving that the subsequence converges to \( L \).

20. Corollary. Let \( (a_n) \) be bounded. Then \( \lim \sup a_n \) is the maximum of the set of subsequential limits.

Proof. The existence of a larger subsequential limit would contradict the eventual upper bound property.

21. Claim. [Sequential Completeness]

Every Cauchy sequence of real numbers converges.

Proof.

Let \( (a_n) \) be Cauchy. Let \( \epsilon > 0 \). There exists \( N \) such that \( |a_n - a_m| < \epsilon \) for all \( n, m \geq N \). In particular, \( |a_n - a_N| < \epsilon \) for all \( n \geq N \).

\[ a_n \text{ for finitely many } n \]

\[ a_N - \epsilon \quad a_N + \epsilon \]

It follows that \( (a_n) \) is bounded and moreover:

\[ a_N - \epsilon \leq \lim \inf a_n \leq \lim \sup a_n \leq a_N + \epsilon \]

Therefore \( \lim \sup a_n - \lim \inf a_n < 2\epsilon \).

Since \( \epsilon \) was arbitrary the limits must be equal proving that \( a_n \) converges.

22. Claim [Cauchy Hadamard]

A (real or complex) power series \[ \sum a_n x^n \] has radius of convergence given by

\[ R = \frac{1}{\lim \sup |a_n|^{1/n}}. \]

Proof. If \[ \lim \sup |a_n|^{1/n} \not\in \{0, \infty\} \]

and \( |x| < R \), then

\[ |x| \lim \sup |a_n|^{1/n} < 1 \]

There exists \( t < 1 \) with

\[ |x| \lim \sup |a_n|^{1/n} < t \]

It follows that for all \( n \) sufficiently large

\[ |x||a_n|^{1/n} \leq t \]

Hence \( |x||a_n|^{1/n} \leq t^n \). Since \( t < 1 \) series converges by comparison with the geometric series. On the other hand, if \( |x| > R \), then

\[ |x| \lim \sup |a_n|^{1/n} > 1 \]

It follows that the

\[ |x||a_n|^{1/n} > 1 \]

for infinitely many terms. Since the \( n \)th term does not tend to 0 the series does not converge. Finally, if the sequence \( \{|a_n|^{1/n}\} \) is unbounded the power series converges only at 0, while if the lim sup is 0 the series converges on the entire plane and \( R = \infty \).

23. Claim. Let \( (a_n) \) be a sequence of positive numbers, then

\[ \lim \inf \left( \frac{a_{n+1}}{a_n} \right) \leq \lim \inf \frac{\sqrt[n]{a_n}}{a_n} \leq \lim \sup \frac{\sqrt[n]{a_n}}{a_n} \leq \lim \sup \left( \frac{a_{n+1}}{a_n} \right) \]

Proof.

By duality it suffices to prove the final inequality. Let \( L = \lim \sup \left( \frac{a_{n+1}}{a_n} \right) \). Let \( A > L \). Then, there exists \( N \) such that
\[
\frac{a_{n+1}}{a_n} < A \text{ for all } n > N. \text{ It follows (by induction) that } a_{N+k} < A^k a_N \text{ for all } k \in \mathbb{N} \text{ and so }
\]
\[
a_n < \left(\frac{a_N}{A^N}\right) A^n, \quad n > N
\]
Hence
\[
\sqrt[n]{a_n} < \left(\frac{a_N}{A^N}\right)^{1/n} A, \quad n > N
\]
Taking the limit superior of both sides and recalling that \(\lim_{n \to \infty} \sqrt[n]{p} = 1\) for all \(p > 0\) gives
\[
\limsup \sqrt[n]{a_n} \leq A.
\]
Since \(A > \limsup \left(\frac{a_{n+1}}{a_n}\right)\) was arbitrary, it follows that
\[
\limsup \sqrt[n]{a_n} \leq \limsup \left(\frac{a_{n+1}}{a_n}\right)
\]
24. Corollary. Let \((a_n)\) be a positive sequence. If \(\lim \frac{a_{n+1}}{a_n} = l\) (where \(l = \infty\) is not excluded) then \(\lim \sqrt[n]{a_n} = l\).

Consider, once more, the (real or complex) power series \(\sum a_n x^n\). The ratio test states that if \((a_n)\) is a sequence with \(a_n \neq 0\) for all \(n\) sufficiently large and
\[
\lim \left|\frac{a_{n+1}}{a_n}\right| = L
\]
then
(a) if \(L < 1\), the series converges absolutely;
(b) if \(L > 1\), the series diverges; If \(L = 1\) then the test is inconclusive.

If \(L\) does not exist, the test can be refined by noting that
(a) if \(\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1\), the series converges absolutely;
(b) if \(\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1\), the series diverges.

26. Remark: If the limit
\[
L = \lim \left|\frac{a_{n+1}}{a_n}\right|
\]
exists then the radius of convergence of power series \(\sum a_n x^n\) is also given by \(R = 1/L\). If \(L\) does not exist it might happen that
\[
\frac{1}{L} < \frac{1}{\limsup \sqrt[n]{|a_n|}} = R
\]
in which case, the ratio test underestimates the radius of convergence. For example, if
\[
a_n = \begin{cases} 2^n & \text{if } n \text{ even,} \\ \frac{1}{2} \cdot 2^n & \text{if } n \text{ is odd.} \end{cases}
\]
then (exercise) \(\sum a_n x^n\) has radius of convergence \(R = \frac{1}{2}\) yet
\[
\limsup \left|\frac{a_{n+1}}{a_n}\right| = 4
\]
which merely guarantees convergence for \(|x| < 1/4\)