Limit superior and limit inferior

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1. Definition. The *limit superior* of a sequence (a_n) is the extended real number defined by

$$\limsup a_n = \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right)$$

Similarly, the *limit inferior* of (a_n) is defined by

$$\liminf a_n = \lim_{n\to\infty}\left(\inf_{k\geq n} a_k\right)$$

2. Existence: Let

$$T_i = \{a_k : k \geq i\}$$

be the i'th tail. Notice that

$$T_1 \supseteq T_2 \supseteq \cdots$$

Hence $\sup T_m$ is always non-increasing and $\inf T_m$ non-decreasing. It follows that

$$\begin{aligned}\limsup a_n &= \lim_{n\to\infty} \sup T_n \\ &= \inf_n (\sup T_n)\end{aligned}$$

and

$$\begin{aligned}\liminf a_n &= \lim_{n\to\infty} (\inf T_n) \\ &= \sup_n (\inf T_n)\end{aligned}$$

always exist as extended real numbers and that

$$\liminf a_n \leq \limsup a_n \tag{1}$$

3. Remark: For all $m, n \in \mathbb{N}$:

$$\begin{aligned}\inf T_n &\leq \inf T_{\max\{m,n\}} \\ &\leq \sup T_{\max\{m,n\}} \\ &\leq \sup T_m\end{aligned}$$

which also implies equation (1).

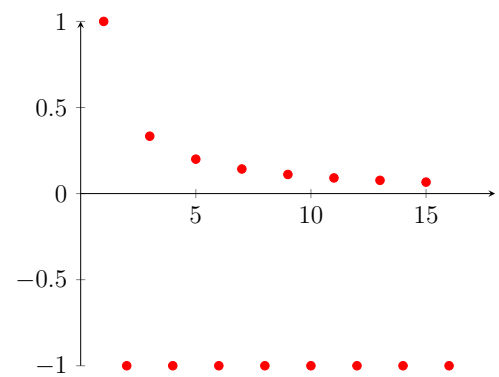
4. Notation.

$\limsup a_n$ and $\liminf a_n$ are often denoted $\overline{\lim}$ and $\underline{\lim}$ respectively.

5. Example.

Let

$$a_n = \begin{cases} \dfrac{1}{n} & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$$



Then $\sup T_n = 1/n$ and $\inf T_n = -1$. Hence

$$\begin{aligned}\limsup a_n &= 0 \\ \liminf a_n &= -1.\end{aligned}$$

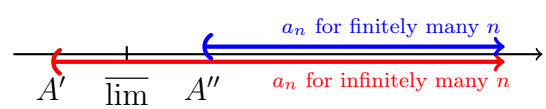
6. Claim. Let (a_n) be a bounded sequence and $A', A'' \in \mathbb{R}$ such that

$$A' < \limsup a_n < A''$$

then

(a) $a_n > A''$ for (at most) finitely many n.

(b) $a_n > A'$ for infinitely many n.



Proof. Exercise

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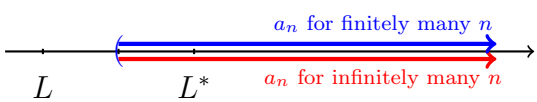
7. Alternative characterization of the limit superior.

   Let (a_n) be a sequence. Then $\limsup a_n$ is the unique extended real number number L with the property that for all $A', A'' \in \mathbb{R}$ with $A' < L < A''$, the following hold:

   (a) $a_n > A''$ for (at most) finitely many n.

   (b) $a_n > A'$ for infinitely many n.

   Proof. The previous claim shows that $L = \limsup a_n$ satisfies the property. If the property is also true for $L^* \neq L$ it is an easy exercise to se that an obvious contradiction arises:



8. Corollary. The limit superior of (a_n) is the smallest real number L such that, for any positive real number ϵ there exists a natural number N such that $a_n < L + \epsilon$ for all $n > N$

   Proof. Exercise.

9. Dual property.

   $\liminf a_n$ is the unique extended real number L such that for all $A', A'' \in \mathbb{R}$ with $A' < L < A''$, the following hold:

   (a) $a_n < A'$ for (at most) finitely many n.

   (b) $a_n < A''$ for infinitely many n.

10. Corollary. The limit inferior of (a_n) is the largest extended real number L such that, for any positive real number ϵ there exists a natural number N such that $a_n > L - \epsilon$ for all $n > N$.

11. Definition. Let $\alpha = (a_n)$ be a bounded sequence. An element $b \in \mathbb{R}$ is an *eventual upper bound* of a α if $b < a_n$ for at most finitely many n.

12. Let B_α be the set of all eventual upper bounds of α. Then

    (a) B_α contains all upper bounds of α
    $$UB(\alpha) \subseteq B_\alpha$$

    (b) B_α is absorbing from the right, i.e.,
    $$x > b \in B_\alpha \Rightarrow x \in B_\alpha$$

    (c) $\inf B_\alpha = \limsup a_n$

    Proof of (c).
    Let $B^* = \inf B_\alpha$ and suppose that $A', A'' \in \mathbb{R}$ satisfy
    $$A' < B^* < A''$$
    Then there exists $b \in B_\alpha$ such that $B^* \leq b < A''$. It follows by right absorption that $A'' \in B_\alpha$ and so $A'' < a_n$ for at most finitely many n. On the other hand
    $$\begin{aligned} & A' < \inf B_\alpha \\ \Rightarrow\ & A' \notin B_\alpha \\ \Rightarrow\ & A' < a_n \text{ for infinitely many } n \end{aligned}$$
    Thus B^* satisfies the characterization of the limit superior in (7). Thus the limit superior of a sequence is the infimum of the set of eventual upper bounds.

13. Warning. Some sources describe the limit superior as an "eventual" upper-bound This terminology is misleading. For example, $\limsup \frac{1}{n} = 0$ yet 0 is a lower bound for the sequence.

14. Exercise. Use the alternative characterizations of limit superior and limit inferior to give another proof of (1).

15. If $\lim a_n$ exists then
    $$\begin{aligned} \liminf a_n &= \lim a_n \\ &= \limsup a_n \end{aligned}$$

    Proof.
    Notice the $\lim a_n$ satisfies the alternative characterizations of the limits superior and inferior given in (7) and (8).

16. If $\liminf a_n = \limsup a_n$ then the sequence (a_n) converges to their common value.

    Proof. From (7), (8) above, it follows that any open interval containing the common value contains all but a finite number of terms of the sequence.

17. Claim.

$$\limsup(a_n+b_n) \leq \limsup(a_n)+\limsup(b_n) \tag{2}$$

Proof: Recall that for any two sequences (a_n and b_n

$$\sup_n\{a_n + b_n\} \leq \sup_n a_n + \sup_n b_n.$$

In particular, applying this to the n-th tails of the two sequences

$$\sup_{k\geq n}\{a_k + b_k\} \leq \sup_{k\geq n} \alpha + \sup_{k\geq n} \beta.$$

Since this holds for all n the limit can be taken on both sides:

$$\begin{aligned}\limsup(a_n + b_n) &= \lim_{n\to\infty}\left(\sup_{k\geq n}(a_k + b_k)\right)\\ &\leq \lim_{n\to\infty}\left(\sup_{k\geq n}(a_k) + \sup_{k\geq n}(b_k)\right)\\ &= \limsup(a_n) + \limsup(b_n)\end{aligned}$$

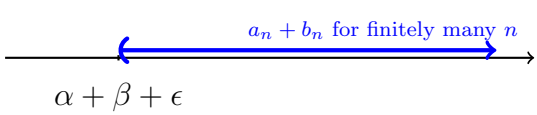
Alternative Proof. Let $\alpha = \limsup a_n$ and $\beta = \limsup b_n$. Then $\exists N, N'$ such that

$$n > N \Rightarrow a_n < \alpha + \epsilon/2$$

$$n > N' \Rightarrow b_n < \beta + \epsilon/2$$

and so

$$n > N + N' \Rightarrow a_n + b_n < \alpha + \beta + \epsilon$$



It must be that

$$\limsup(a_n + b_n) \leq \alpha + \beta + \epsilon.$$

Since ϵ was arbitrary

$$\limsup(a_n + b_n) \leq \alpha + \beta$$

and the result follows.

18. Mean Property

    Let (a_n) be a bounded sequence. Define (x_n) by

$$x_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

Then

$$\limsup x_m \leq \limsup a_n$$

with an obvious dual statement for the limit inferior.

Proof. Let $\epsilon > 0$. Then, there exists N such that

$$n > N \Rightarrow x_n < \limsup a_n + \epsilon$$

Let $U = \limsup a_n$ and $n > N$.Then

$$x_n < \frac{a_1 + a_2 + \cdots + a_N + (n - N)[U + \epsilon]}{n} \tag{3}$$

Notice that for all n sufficiently large,

$$x_n < U + \epsilon$$

This implies that $\limsup x_n \leq U$. (Remark: This also follows by applying property (2) to equation (3).

19. Bolzano Weierstrass Theorem.

    A bounded convergent sequence has a convergent subsequence.

    Proof. Let (a_n) be bounded. The idea is to construct a subsequence converging to $L = \limsup a_n$.

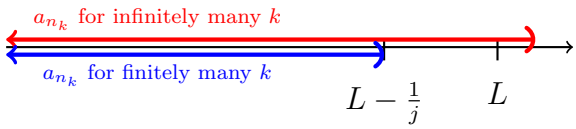
    $\exists n_1$ such that $a_{n_1} > L - 1$. Proceeding by recursion, suppose that there exist natural numbers

$$n_1 < n_2 < \cdots < n_k$$

such that

$$a_{n_j} > L - 1/j, \quad 1 \le j \le k$$

It is not assumed that the a_{n_j} are increasing. However, since $a_{n_k} < L - 1/j$ for finitely many n_k



it follows that $L \le \liminf a_{n_k}$ and so

$$L \le \liminf a_{n_k} \le \limsup a_{n_k} \le L$$

proving that the subsequence converges to L.

20. Corollary. Let (a_n) be bounded. Then $\limsup a_n$ is the maximum of the set of subsequential limits.

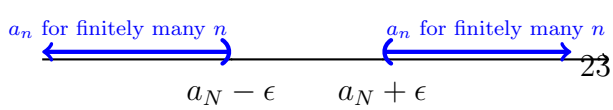
    Proof. The existence of a larger subsequential limit would contradict the eventual upper bound property.

21. Claim. [Sequential Completeness]

    Every Cauchy sequence of real numbers converges.

    Proof.

    Let (a_n) be Cauchy. Let $\epsilon > 0$. There exists N such that $|a_n - a_m| < \epsilon$ for all $n, m \ge N$. In particular, $|a_n - a_N| < \epsilon$ for all $n \ge N$.



    It follows that (a_n) is bounded and moreover:

$$a_N - \epsilon \le \liminf a_n \quad \le \limsup a_n \le a_N + \epsilon$$

    Therefore $\limsup a_n - \liminf a_n < 2\epsilon$. Since ϵ was arbitrary the limits must be equal proving that a_n converges.

22. Claim [Cauchy Hadamard]

    A (real or complex) power series $\sum a_n x^n$ has radius of convergence given by

$$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}.$$

    Proof. If

$$\limsup(|a_n|^{1/n} \notin \{0, \infty\}$$

    and $|x| < R$, then

$$|x| \limsup |a_n|^{\frac{1}{n}} < 1$$

    There exists $t < 1$ with

$$|x| \limsup |a_n|^{\frac{1}{n}} < t$$

    It follows that for all n sufficiently large

$$|x||a_n|^{\frac{1}{n}} \le t$$

    Hence $|a_n||x|^n \le t^n$. Since $t < 1$ series converges by comparison with the geometric series. On the other hand, if $|x| > R$, then

$$|x| \limsup |a_n|^{\frac{1}{n}} > 1$$

    It follows that the

$$|x||a_n|^{\frac{1}{n}} > 1$$

    for infinitely many terms. Since the nth term does not tend to 0 the series does not converge. Finally, if the sequence $(|a_n|^{1/n})$ is unbounded the power series converges only at 0, while if the lim sup is 0 the series converges on the entire plane and $R = \infty$.

23. Claim. Let (a_n) be a sequence of positive numbers, then

$$\begin{aligned} \liminf \left(\frac{a_{n+1}}{a_n}\right) &\le \liminf \sqrt[n]{a_n} \\ &\le \limsup \sqrt[n]{a_n} \\ &\le \limsup \left(\frac{a_{n+1}}{a_n}\right) \end{aligned}$$

    Proof.

    By duality it suffices to prove the final inequality. Let $L = \limsup \left(\frac{a_{n+1}}{a_n}\right)$. Let $A > L$. Then, there exists N such that

$\frac{a_{n+1}}{a_n} < A$ for all $n > N$. It follows (by induction) that $a_{N+k} < A^k a_N$ for all $k \in \mathbb{N}$. and so

$$a_n < \left(\frac{a_N}{A^N}\right) A^n, \quad n > N$$

Hence

$$\sqrt[n]{a_n} < \left(\frac{a_N}{A^N}\right)^{1/n} A, \quad n > N$$

Taking the limit superior of both sides and recalling that $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$ for all $p > 0$ gives

$$\limsup \sqrt[n]{a_n} \leq A.$$

Since $A > \limsup \left(\frac{a_{n+1}}{a_n}\right)$ was arbitrary, it follows that

$$\limsup \sqrt[n]{a_n} \leq \limsup \left(\frac{a_{n+1}}{a_n}\right)$$

24. Corollary. Let (a_n) be a positive sequence. If $\lim \frac{a_{n+1}}{a_n} = l$ (where $l = \infty$ is not excluded) then $\lim \sqrt[n]{a_n} = l$.

25. Refinement of d'Alembert's ratio test. Consider, once more, the (real or complex) power series $\sum a_n x^n$. The ratio test states that if (a_n) is a sequence with $a_n \neq 0$ for all n sufficiently large and

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = L$$

then

(a) if $L < 1$, the series converges absolutely;

(b) if $L > 1$, the series diverges; If $L = 1$ then the test is inconclusive.

If L does not exist, the test can be refined by noting that

(a) if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series converges absolutely;

(b) if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, the series diverges.

26. Remark: If the limit

$$L = \lim \left|\frac{a_{n+1}}{a_n}\right|$$

exists then the radius of convergence of power series $\sum a_n x^n$ is also given by $R = 1/L$. If L does not exist it might happen that

$$\frac{1}{L} < \frac{1}{\limsup \sqrt[n]{|a_n|}} = R$$

in which case, the ratio test underestimates the radius of convergence. For example, if

$$a_n = \begin{cases} 2^n & \text{if } n \text{ even,} \\ \frac{1}{2} \cdot 2^n & \text{if } n \text{ is odd.} \end{cases}$$

then (exercise) $\sum a_n x^n$ has radius of convergence $R = \frac{1}{2}$ yet

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| = 4$$

which merely guarantees convergence for $|x| < 1/4$