

Math 5201 - Homework on the Real Numbers

Philip Pennance¹ – Semester II, 2012-13.

1. Let E be an equivalence relation on a set V . Show that the set of equivalence classes of E is a partition of V .
2. Show that the null Cauchy sequences form an ideal in the ring of Cauchy sequences.
3. Let J be the ideal of null Cauchy sequences. For two Cauchy sequences α and β . Define $\alpha \sim \beta$ if $\alpha - \beta \in J$.
 - (a) Show that $\sim$ is an equivalence relation.
 - (b) Suppose $\alpha \sim \beta$ and $\gamma \sim \delta$. Prove $\alpha + \gamma \sim \beta + \delta$.
 - (c) Let $[\alpha]$ denote the equivalence class of a Cauchy sequence α . Define $[\alpha] \oplus [\beta] = [\alpha + \beta]$. Show that $\oplus$ is well defined.
4.
 - (a) State the Archimedean property.
 - (b) Let $\epsilon > 0$ and $y \in \mathbb{R}$. Show that there exists a natural number n such that $n\epsilon > y$.
 - (c) Using $\epsilon - N$ arguments and the Archimedean property, show that the sequences $(\frac{1}{n})$ and $(\frac{1}{n-1})$ converge.
 - (d) Show carefully that the sequence $((-1)^n)$ does not converge.
5. Let $S = \{q \in \mathbb{Q} : q^2 < 2\}$.
 - (a) Show that $M \in \mathbb{Q}$ is an upper bound for S if and only if $M^2 > 2$.
 - (b) Show that the set S has no maximum element.
 - (c) Show that S has no least upper bound in $\mathbb{Q}$.
6. Let R' and R be ordered rings. Let P' and P respectively, denote their sets of positive elements. Let $f : R' \rightarrow R$ be an embedding (i.e. an injective homomorphism). Show that f is order preserving if and only if $f^{-1}(P) = P'$.
7. Let R be an ordered ring and let $x, y \in R$. Show that:
 - (a) $|x - y| \geq |x| - |y|$
 - (b) $|x + y| \geq |x| - |y|$
8. Let R be an ordered ring. If $\xi \in R$ satisfies $0 \leq \xi < \epsilon$ for all positive $\epsilon \in R$, show that $\xi = 0$.
9. Let K be an ordered field and
$$f : \mathbb{Q} \rightarrow K$$
an embedding of the rationals into K . Show that f is order preserving.

¹<https://pennance.us>