# Connectedness.

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1. Definition.

   A topological space $X$ is *connected* if it cannot be divided into two disjoint, nonempty, open sets.

   A subset of a topological space is said to be *connected* if it is connected under its subspace topology.

2. Claim.

   $I = [0, 1]$ is connected.

   Proof.

   Suppose to the contrary that $I$ is not connected. Then

   $$I = A \dot{\cup} B$$

   where $A, B$ are open (in the subspace topology) and non empty. It can be assumed that $0 \in A$. Let

   $$T = \{\tau : [0, \tau] \subseteq A\}$$

   Since $A$ is non empty and bounded above, it has a supremum $\xi$. Since $A, B$ are disjoint and non empty, it must be that $\xi < 1$. If $\xi \in A$, then since $A$ is open, there exists $\delta > 0$ such that $\xi + \delta \in A$ contradicting the fact that $\xi$ is an upper bound for $T$. If $\xi \in B$, since $B$ is open, there exists $\delta > 0$ such that $\xi - \delta\} \in B$ again contradicting the definition of $\xi$.

3. Claim.

   A path-connected space $X$ is connected.

   Proof.

   Suppose $X$ is not connected, so that there exist $A, B$ disjoint nonempty such that $X = A \dot{\cup} B$. Let $a \in A$ and $b \in B$. Let $\gamma : [0, 1] \to X$ be an $(a, b)$-path with $\gamma(0) = a$ and $\gamma(1) = b$. Then, it is easy to see that

   $$[0, 1] = \gamma^{-1}[A] \dot{\cup} \gamma^{-1}[b]$$

   Since $\gamma$ is continuous, the (nonempty) sets $\gamma^{-1}[A]$ and $\gamma^{-1}[b]$ are open. This contradicts the connectedness of the interval $[0, 1]$.

4. Corollary.

   Open disks in $\mathbb{C}$ are connected.

5. Definition.

   Two subsets $A$ and $B$ of a topological space are *separated* if they are disjoint and neither contains an accumulation point of the other.

6. Lemma.

   Let $A$ and $B$ be connected subsets of a topological space. If $A$ and $B$ are not separated then $A \cup B$ is connected.

   Proof.

   Suppose to the contrary that $A \cup B$ is not connected. Since both $A$ and $B$ are connected, it must be (exercise) that $A \cup B$ is a disjoint union. Without loss of generality, suppose that some point $a \in A$ belongs to $B'$. Then every neighborhood of $a$ intersects $B$. This is impossible since $A$ is open and $A$ and $B$ are disjoint.

7. Claim.

   Connectedness does not imply path connectedness.

   Counter example: Let $\mathcal{L}$ be the set of lines

   $$y = \frac{1}{n}x, \quad n \in \mathbb{N}$$

   Let $A = \cup\mathcal{L}$ and let $B = \{(0, 1)\}$. Then $A$ and $B$ are connected. Also, $A$ and $B$

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are not separated since $(0,1) \in A'$. It follows that $A \cup B$ is connected. Let $p : [0,1] \to \mathcal{L}$ be a path such that p(0) = (1,0). It can be shown by a technical argument that that $p(t) = (1,0)$ for all $t \in [0,1]$. From this it follows that $A \cup B$ is not path connected (which is intuitively obvious).

8. Claim.

   Let $A \subseteq \mathbb{C}$ be open. If $A$ is connected, then $A$ is path connected.

   Proof.

   Let $z_0 \in A$ and

   $$A_1 = \{z \in A : \text{ there exists an } (z_0, z)\text{-path}\}$$

   the set of elements *reachable* from $z_0$. It suffices to show that $A_1 = A$. Suppose to the contrary that $A_1 \neq A$. Then $A_2 = A \setminus A_1 \neq \emptyset$. Let $z \in A_1$. since $A$ is open, there exists an open disk $D$ such that $z \in D \subseteq A$. Since $z \in D$ every point in $D$ is reachable so $z \in D \subseteq A_1$. That is, $A_1$ is open.

   We claim that $A_1$ is also closed. Let $a$ be an accumulation point of $A_1$. Then any open disk containing $a$ must contain a point of $A_1$ different from $a$. It follows that $a$ is reachable. Therefore, $A_1$ is also closed and $A_2$ is an open set. This is a contradiction since $A = A_1 \dot{\cup} A_2$ would then be a disconnection of $A$.