

Bondy–Chvátal Closure and Hamiltonicity¹

1. Lemma. If $G \neq K_2$ has a Hamiltonian path $(a = v_1, v_2, \dots, v_n = b)$ and

$$d(a) + d(b) \geq |V(G)|$$

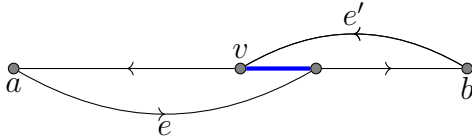
then G has a Hamiltonian cycle containing all edges of P except one.

Proof. If a, b are adjacent, and $G \neq K_2$ then $P + ab$ is a Hamiltonian cycle. If a, b are not adjacent let

$$A = \{v_i \in V : \{a, v_{i+1}\} \in E(G)\}.$$

$$B = \{v_k \in V : \{v_k, b\} \in E(G)\}.$$

Notice that B consists of the neighbors of b which belong to P whereas A consists of left translates along the path of neighbors of a . Therefore $|A| = d(a)$ and $|B| = d(b)$. Suppose first that there exists $v \in A \cap B$. Then, there exists an obvious Hamiltonian cycle, as depicted below. This cycle avoids the edge drawn in blue.



If, on the other hand, $A \cap B = \emptyset$, then

$$\begin{aligned} |A \cup B| &= |A| + |B| \\ &= d(a) + d(b) \\ &\geq |V(G)| \end{aligned}$$

which is impossible since $b \notin A \cup B$.

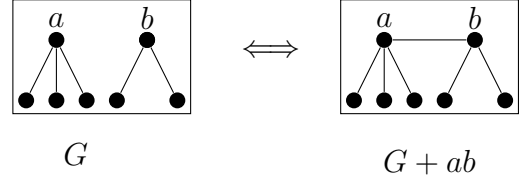
Claim [Bondy and Chvátal, 1974.]

If $a, b \in V(G)$ are not adjacent and

$$d(a) + d(b) \geq |V(G)| \quad (1)$$

the following are equivalent

- (a) G has a Hamiltonian cycle.
- (b) $G + \{a, b\}$ has a Hamiltonian cycle.



Proof. If G has a Hamiltonian cycle C , then trivially, C is a Hamiltonian cycle of $G + ab$. Conversely, suppose that $G + ab$ has a Hamiltonian cycle C . If $ab \notin C$ then C is a Hamiltonian cycle of G . If $ab \in C$ then $C \setminus ab$ is a Hamiltonian path of G . Using (1) and the lemma, it follows that G has a Hamiltonian cycle.

2. Claim. The output of the following algorithm is unique.

Algorithm: $cl(G)$

Input: A graph G

Output: A graph $cl(G)$ that contains G as a spanning subgraph

Let $G_1 = G$;

while there exist non-adjacent vertices $a, b \in V(G_i)$ such that $d(a) + d(b) \geq |V(G_i)|$ **do**

$G_{i+1} \leftarrow G_i + (a, b);$
 $i \leftarrow i + 1;$

return G_i ;

Proof. The claim is that $G \mapsto cl(G)$ is a function. Suppose, to the contrary that the algorithm produces graphs F and H with $F \neq H$.

$$F = G + \langle f_1, f_2, \dots, f_n \rangle$$

¹Notes based on Lectures by A. Kelmans at the University of Puerto Rico, 1996. All errors are mine.

$$H = G + \langle h_1, h_2, \dots, h_m \rangle$$

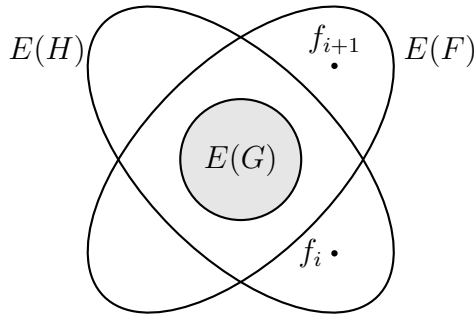
Let f_i be the first edge in F but not in H . Let $L = G + \langle f_1, f_2, \dots, f_{i-1} \rangle$. Notice that L is a subgraph of both F and H . Suppose that $f_i = \{a, b\}$. By the algorithm,

$$d_L(a) + d_L(b) \geq |V(G)|.$$

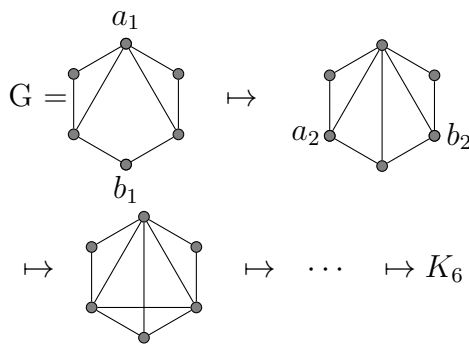
Also

$$d_H(a) + d_H(b) \geq d_L(a) + d_L(b) \geq |V(G)|.$$

Hence $\{a, b\}$ is an H admissible edge contradicting the algorithm.



3. Example.



4. Corollary 1. [Bondy and Chvátal] G is Hamiltonian if and only if $cl(G)$ is Hamiltonian .

5. Corollary 2. [Dirac 1952]

If $\nu \geq 3$ and $\delta \geq \nu/2$ then G is Hamiltonian .

6. Corollary 3. If $n > 2$ and $cl(G) = K_n$, then G is Hamiltonian .

7. Corollary 4. [Ore, 1960]. If $\nu \geq 3$ and for all non adjacent vertices a, b

$$d(a) + d(b) \geq \nu$$

then G is Hamiltonian .

8. Exercise. If $n \neq m$ show that $K_{n,m}$ is not Hamiltonian. Find an (A,B)-bipartite graph with $|A| = |B|$ which has no Hamiltonian cycle.

9. Show that every graph that satisfies Dirac's condition automatically satisfies Ore's condition.

10. Exercise. Suppose that there exists an oracle HP which solves the Hamiltonian path problem (I.e., given a graph G and two vertices $x, y \in V(G)$, find a (x, y) -Hamiltonian path or return NO if no such path exists.) Given HP, can we solve the Hamiltonian cycle problem in a polynomial number of calls to the oracle HP?

Solution.

Apply HP to each edge e of G . If HP returns a Hamiltonian path P , then $P + e$ is a Hamiltonian cycle. If HP returns NO for every edge, then there is no Hamiltonian path and so there can be no Hamiltonian cycle.

11. Exercise. Suppose that there exists an oracle HC which given a graph G either returns a Hamiltonian cycle or returns NO if none exists. Given HC, can we solve the Hamiltonian path problem in a polynomial number of calls to HC?

Solution.

Let $x, y \in V(G)$.

Case 1. If x, y are not adjacent, notice that there exists a Hamiltonian path in G if and only if there exists a Hamiltonian cycle in $G + xy$ so a single call to the oracle with the graph $G + xy$ solves the Hamiltonian path problem.

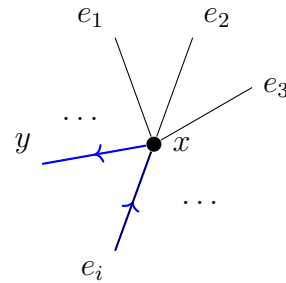
Case 2. If x, y are adjacent then there exists an (x, y) -Hamiltonian path in G if and only if there exists a Hamiltonian cycle in $G + xy$ containing xy . Notice that a Hamiltonian cycle containing xy must contain exactly one other edge e_i incident to x . Let

$$S(x) = \{e \in E(G) : e \text{ is incident to } x\}$$

(the star at x). For each $e_i \in S(x)$ consider the graph

$$G \setminus S(x) + xy + e_i$$

and submit it to HC. If, for some i , there exists a Hamiltonian cycle, then there exists a Hamiltonian path xPy in G . If for all i no such Hamiltonian cycle exists, then there does not exist a Hamiltonian path in G .



$S(x)$ - The star at x