

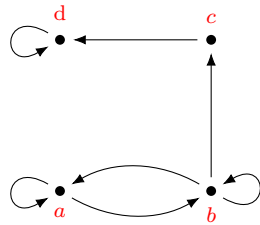
## Lecture Summary - Introduction to Digraphs

1. A *directed graph* is a pair  $(V, E)$  where  $V$  is a non empty set of elements called *vertices* and  $E$  is a binary relation on  $V$ , (i.e.  $E$  is a subset of  $V \times V$ ). The elements of  $E$  are called *directed edges*.
2. Example. Let

$$V = \{a, b, c, d\}$$

$$E = \{(a, a), (b, b), (d, d), (a, b), (b, a), (b, c), (c, d)\}$$

Then  $G = (V, E)$  is a directed graph.  $G$  is depicted below.



3. Notation. Let  $G = (V, E)$  be a directed graph and  $x, y \in V$ . We write  $xEy$  or  $x \rightarrow y$  to indicate that  $(x, y) \in E$ .
4. A binary relation  $E$  on a set  $V$  is called:
  - (a) *reflexive* if  $aEa$  for all  $a \in V$ .
  - (b) *irreflexive* if  $aEa$  for no  $a \in V$ .
  - (c) *symmetric* if  $aEb$  implies  $bEa$ , for all  $a, b \in V$ .
  - (d) *asymmetric*;  
if  $aEb$  and  $bEa$  imply  $a = b$ , for all  $a, b \in V$ .
  - (e) *strictly asymmetric*;  
if  $aEb$  implies  $\neg bEa$ , for all  $a, b \in V$ .
  - (f) *transitive* if  $aEb$  and  $bEc$  imply  $aEc$  for all  $a, b, c \in V$ .
  - (g) a *pre-order* if it is reflexive and transitive.

- (h) an *order* if it is reflexive, transitive and asymmetric;
- (i) a *strict order* if it is transitive and strictly asymmetric;
- (j) an *equivalence relation* if reflexive, transitive and symmetric.

5. Associated with any pre-order are the following relations:

- (a) *less than*

$$a < b \Leftrightarrow aEb \text{ and not } bEa$$

- (b) *greater than*

$$a > b \Leftrightarrow bEa \text{ and not } aEb$$

- (c) *incompatible with*

$$a \parallel b \Leftrightarrow \text{neither } aEb \text{ nor } bEa$$

- (d) *equivalent to*

$$a \sim b \Leftrightarrow aEb \text{ and } bEa$$

6. Claim. Let  $G = (V, E)$  be a directed graph. If  $E$  is a preorder (reflexive and transitive) then

- (a)  $\sim$  is an equivalence relation.
- (b)  $>$  and  $<$  are strict orders.
- (c) If  $E$  is an equivalence relation then  $\sim$  and  $E$  coincide and neither  $a < b$  nor  $b < a$  is possible by symmetry. Hence, if  $E$  is an equivalence relation, we will often write  $a \sim b$  instead of  $aEb$ .
- (d) If  $E$  is an order relation then  $a \sim b$  is possible only if  $a = b$ . Hence, if  $E$  is an order relation, we often write  $\preceq$  instead of  $aEb$ .

7. Definition.

An order relation  $\preceq$  on a set  $V$  is called *total* if either  $a \preceq b$  or  $b \preceq a$  for all  $a, b \in V$ .

8. If  $E$  is an preorder relation, we will write  $\lesssim$  instead of  $aEb$ .

9. Exercise. Let  $\lesssim$  be a preorder. Define a relation  $\preceq$  on the set  $V/\sim$  of equivalence classes of the associated relation  $\sim$  by

$$[x] \preceq [y] \Leftrightarrow x \lesssim y.$$

Then:

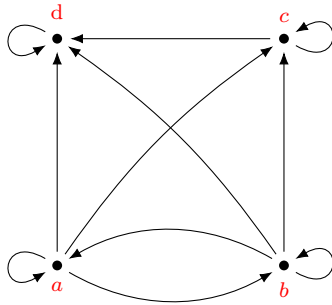
- (a)  $\preceq$  is well defined. (i.e.) If  $x' \in [x]$  and  $y' \in [y]$  and  $x \lesssim y$ , then  $x' \lesssim y'$  and so the relation is independent of equivalence class representative.
- (b)  $\preceq$  is an order,  
(called the order *induced by*  $\lesssim$ .)

Proof of antisymmetry:

If  $[x] \preceq [y]$  and  $[y] \preceq [x]$  then  $x \lesssim y$  and  $y \lesssim x$  so we have  $x \sim y$ . Since  $\sim$  is an equivalence relation it follows that  $[x] = [y]$ .

10. Example.

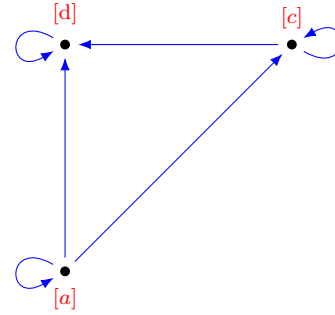
Let  $G = (V, E)$  be the preorder drawn below.



The set  $V/\sim$  of equivalence classes of the relation  $\sim$  is

$$\begin{aligned} V/\sim &= \{\{a, b\}, \{c\}, \{d\}\} \\ &= \{[a], [c], [d]\} \end{aligned}$$

The order  $(V/\sim, \preceq)$  is drawn below.



11. Let  $(V, \lesssim)$  a preorder and let  $A$  be a subset of  $V$ . An element  $a \in A$  is called:

- (a) a *minimal* element of  $A$  if there does not exist  $a' \in A$  such that  $a' < a$ .
- (b) a *maximal* element of  $A$  if there does not exist  $a' \in A$  such that  $a' > a$ .
- (c) a *minimum* (least) element of  $A$  if  $a \lesssim x$  for every  $x \in A$ .
- (d) a *maximum* element of  $A$  if  $x \lesssim a$  for every  $x \in A$ .

12. Remark. If  $a$  is a minimum of  $A$  and  $b \sim a$  then  $b$  is also a minimum. Thus a minimum is unique up to equivalence (and is unique when  $\lesssim$  is an order). The same holds for maxima. Every minimum is minimal, but not conversely.

13. Let  $(V, \lesssim)$  a preorder and let  $A$  be a subset of  $V$ . An element  $v \in V$  is called:

- (a) a lower bound of  $A$  if  $v \lesssim x$  for all  $x \in A$ .
- (b) an upper bound for  $A$  if  $x \lesssim v$  for all  $x \in A$ .
- (c) a greatest lower bound (glb) or *infimum* of  $A$  if  $v$  is a lower bound of  $A$  and in addition  $v' \lesssim v$  for every other lower bound  $v'$  of  $A$ .
- (d) a least upper bound (lub) or *supremum* of  $A$  if  $v$  is an upper bound of  $A$  and in addition  $v \lesssim v'$  for every other upper bound  $v'$  of  $A$ .

## 14. Remarks/exercises:

- (a) Lower and upper bounds of a subset  $A$  are not necessarily elements of  $A$
- (b) Minimal, maximal, minimum, and maximum elements of  $A$  must be elements of  $A$ .
- (c) If an ordered set has exactly one maximal element, then that element is not necessarily the greatest.
- (d) If  $v$  and  $v'$  are glb's then (by antisymmetry)  $v \sim v'$ .
- (e) In a total order a minimal element is minimum.

15. Remarks. Let  $(V, \leq)$  be an order and  $A \subseteq V$ .

- (a) An element  $a \in A$  is minimum if for all  $x \in A$ ,  $a \leq x$ .
- (b)  $A$  can have at most one glb and at most one lub. These are denoted  $\text{glb } A$  and  $\text{lub } A$  respectively.
- (c) The greatest lower bound of two elements,  $a$  and  $b$  (if it exists) is also called the *meet*.
- (d) The least upper bound of two elements,  $a$  and  $b$  (if it exists) is also called the *join*.

## 16. Example.

The order  $\{(a, c), (a, b), (a, a), (b, b), (c, c)\}$  on  $\{a, b, c\}$  has 2 maximal elements, neither of which is a maximum.

## 17. Example.

Let  $V = \mathbb{N} \cup \{\alpha\}$  where  $n \parallel \alpha$  for all  $n \in \mathbb{N}$ . Then  $\alpha$  is a unique maximal element in  $V$  yet is not a maximum.

## 18. Example.

Consider the order  $(V, E)$  where  $V = \mathbb{N} \setminus \{0, 1\}$  and  $aEb \Leftrightarrow a|b$ . Then

- (a) Every prime is a minimal element.
- (b) There is no maximal element.

(c) There is no minimum element.

(d) There is no maximum element.

19. An order  $(V, \leq)$  is called a *lattice* if for all  $a, b \in V$ , the set  $\{a, b\}$  has a glb and a lub.

## 20. Example.

Let  $S$  be a finite set. Consider the order  $(V, E)$  where  $V = 2^S$  and for any two sets  $X$  and  $Y$ ,

$$(X, Y) \in E \Leftrightarrow X \subseteq Y$$

- (a)  $S$  is the maximum element of  $V$ .
- (b)  $\emptyset$  is the minimum element of  $V$ .
- (c)  $(V, E)$  is a lattice.

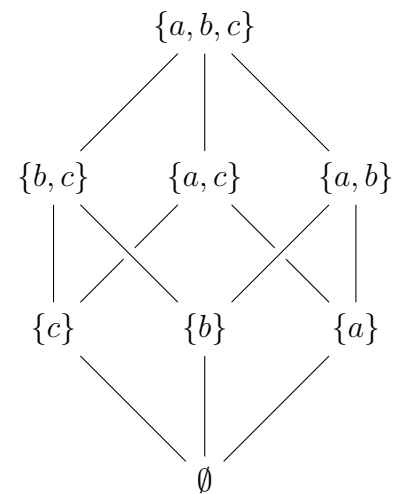
The greatest lower bound (meet) of two subsets  $X, Y \subseteq S$  is their intersection.

$$\text{glb}(A, B) = A \cap B$$

The least upper bound (join) is their union

$$\text{lub}(A, B) = A \cup B$$

A Hasse diagram for the case  $S = \{a, b, c\}$  is drawn below.



## 21. Definition.

Let  $a, b \in \mathbb{N}$ . The *least common multiple* of  $a$  and  $b$  (denoted  $\text{lcm}(a, b)$ ) is the natural number  $\ell$  which satisfies:

- (a)  $a|\ell$  and  $b|\ell$   
 (b)  $a|n$  and  $b|n \Rightarrow \ell|n$  for all  $n \in \mathbb{N}$

22. Example.

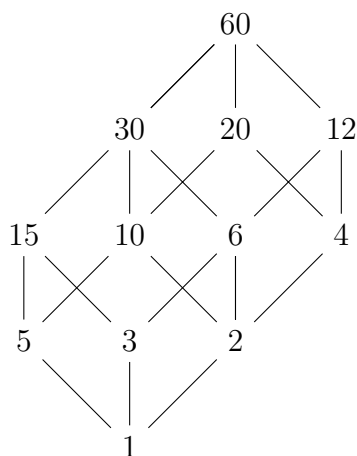
Consider the order  $(V, E)$  where  
 $V = \mathbb{N}$  and  $aEb \Leftrightarrow a|b$ . Then

- (a) 1 is minimum.  
 (b) 0 is maximum.  
 (c)  $(\mathbb{N}, |)$  is a lattice. If  $a, b \in \mathbb{N}$  then

$$\text{glb}(a, b) = \text{gcd}(a, b)$$

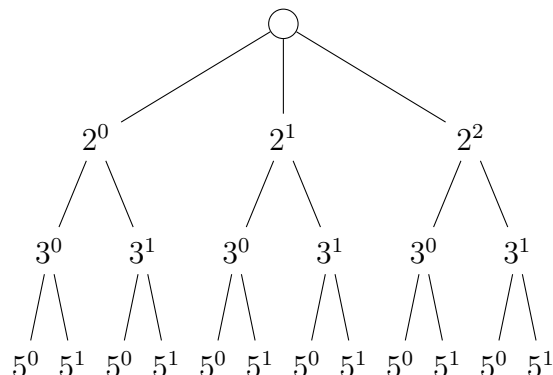
$$\text{lub}(a, b) = \text{lcm}(a, b)$$

23. Example. The factors of 60 form a lattice under the divisibility relation  $aEb \Leftrightarrow a|b$ .



Remark:

Since  $60 = 2^2 3^1 5^1$ , the factors of 60 are in one to one correspondence with paths from the root to the leaves of the tree:



Hence 60 has

$$|\{0, 1, 2\} \times \{0, 1\} \times \{0, 1\}| = 12$$

factors, specifically

$$\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$