

Trees and Forests¹

1. A graph T is a *tree* if it is

- (a) connected,
- (b) acyclic (i.e., has no cycles).

A graph is called a *forest* if it has no cycles.

2. Corollary. Each connected component of a forest is a tree.
3. Let T be a tree and $e \in E(T)$. Then $T \setminus e$ has two components, and each component of $T \setminus e$ is a tree.

Proof. Let $e = xy \in E(T)$. Since T is connected and acyclic, e is a bridge. Hence $T \setminus e$ has two components. Removal of an edge cannot create a cycle, so each component is a tree.

4. Let T be a forest with $|E(T)| \geq 1$. Then T has at least two leaves (vertices of degree 1).

Proof 1. Consider a longest path in T (exercise).

Proof 2. By induction on $|E(T)|$. If $|E(T)| = 1$, the unique edge has two ends of degree 1, so T has at least two leaves.

Suppose the result holds for every forest with at most $n - 1$ edges, and let T be a forest with $|E(T)| = n \geq 2$. Fix $e \in E(T)$. Deleting e splits the component of T that contained e into two trees T_1 and T_2 ; the other components of T are unchanged.

- If both T_1 and T_2 have an edge, the induction hypothesis gives each at least two leaves. Restoring e can destroy at most two of those leaves, so T has at least two leaves.

- If one of T_1, T_2 is an isolated vertex and the other has an edge, the latter has at least two leaves by induction. Restoring e destroys at most one of them and creates a leaf at the isolated vertex, so T still has at least two leaves.
- If both T_1 and T_2 are isolated vertices, then e itself is a component of T , and restoring e produces two leaves.

5. Remark. The existence of a leaf is the basis for many induction proofs concerning trees. This fact is formalized in the following result.

6. Let T be a tree. There exist trees $T_0, T_1, \dots, T_N$ such that T_0 is a vertex, $T_N = T$, and for each $0 \leq i \leq N - 1$, the tree T_{i+1} is obtained from T_i by the addition of an edge.

Proof. By induction on the number of vertices. If T is a single vertex, take $T_0 = T$.

Now let T be a tree with $n \geq 2$ vertices, and assume the result holds for every tree with fewer than n vertices. Then T has a leaf x , and $T \setminus x$ is a tree. By the induction hypothesis there is a sequence of trees

$$T_0, T_1, \dots, T_{N-1} = T \setminus x$$

satisfying the requirements of the theorem. Set $T_N = T$. Since T is obtained from $T \setminus x$ by adding the edge incident with x , the sequence $T_0, T_1, \dots, T_N$ has the required property.

¹Notes based on Lectures by A. Kelmans at the University of Puerto Rico, 1996.

7. Let T be a tree and $x, y \in V(T)$. Then there exists a unique xy -path.

Proof. By induction on $|E(T)|$. If $|E(T)| = 0$, then T is a single vertex and the only path is the trivial path at that vertex. If $|E(T)| = 1$, then T is an edge and the claim is immediate.

Now let T be a tree with $|E(T)| > 1$, and assume the result holds for every tree with fewer edges. Let z be a leaf of T , with unique neighbour w . Then $T \setminus z$ is a tree, so the induction hypothe-

sis gives a unique path between any two vertices of $T \setminus z$.

If $x, y \in V(T \setminus z)$, that unique path is also the unique xy -path in T : any xy -path through z would have to enter and leave z , which is impossible since $\deg(z) = 1$.

If one of x, y is z , say $y = z$ and $x \neq z$, then every xz -path in T is an xw -path in $T \setminus z$ followed by the edge wz . The xw -path is unique by induction, so the xz -path is unique.