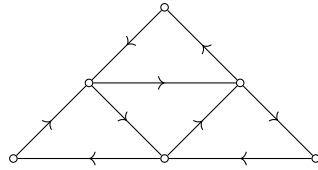


The Last Departure Graph of an Euler Circuit¹

1. A digraph is *balanced* if $d_{\text{in}}(v) = d_{\text{out}}(v)$ for all $v \in V$.

2. Example.



A balanced digraph

3. Let G be directed. Then G has an Euler tour if and only if G is connected and balanced.

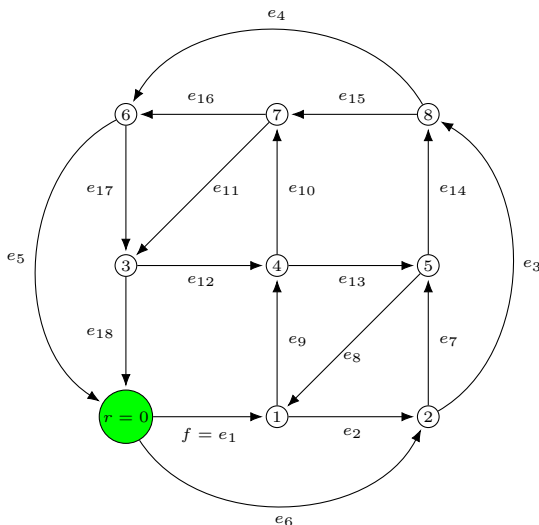
Proof: similar to the undirected case.

4. Let G be directed and balanced. An Euler circuit

$$Q = \langle r, f, v_1, e_1, \dots, e_{m-1}, r \rangle$$

with root r and first edge f induces an ordering of the edges incident with any vertex $x \in V(G)$.

5. Example. The following is a drawing of a balanced digraph and a Euler tour Q with initial vertex $r = 0$ and first edge e_1



6. The *last departure edge* of an Euler tour Q at a vertex x will be denoted by $\ell_Q(x)$.

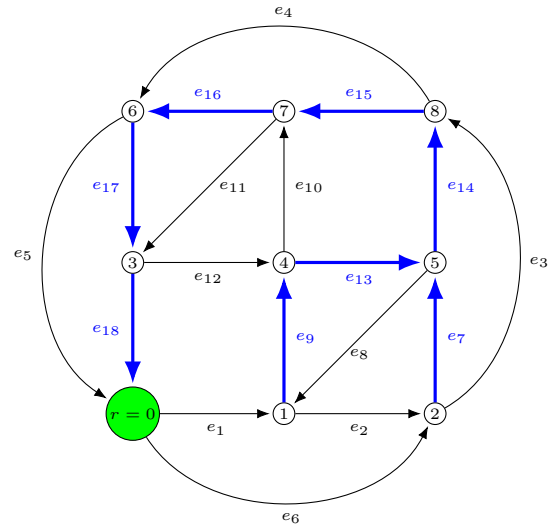
7. The *last departure graph* of an Euler circuit Q , is the subgraph induced by the edge set

$$L_Q = \{\ell_Q(x) : x \in V(G) \setminus \{r\}\}$$

Thus if $LD(Q)$ is the last departure graph of an Euler tour Q then

$$E(LD(Q)) = L_Q; \quad V(LD(Q)) = V(G).$$

8. Example. The last departure graph $LD[Q]$ of the indicated Euler tour Q is shown in blue.



9. Let x be a vertex of $LD[Q]$. Then

$$d_{\text{out}}(x) = \begin{cases} 1 & \text{if } x \neq r, \\ 0 & \text{if } x = r. \end{cases}$$

10. Claim.

Let $G = (V, E)$ be a directed graph with Euler tour Q . Then $LD[Q]$ is an arborescence: a tree with all edges directed toward a root of out-degree zero.

Proof. Since $d_{\text{out}}(x) \leq 1$, every component of $LD[Q]$ is a directed tree terminating at a cycle or at a vertex. Suppose, for a contradiction, that $LD[Q]$ has a cycle

¹Notes based on Lectures by A. Kelmans at the University of Puerto Rico, 1996. All errors are mine.

$C = \langle e_1, e_2, \dots, e_n \rangle$. The Euler circuit Q induces a partial order on C : for $a, b \in E(C)$, write $a \preceq b$ if Q uses a before b . Let $e_1 = (v_1, v_2) \in C$. If e_1 and e_2 are consecutive on Q , then $e_1 \preceq e_2$. If not, then after e_1 the tour leaves v_2 by some $u \neq e_2$, and later uses e_2 , so again $e_1 \preceq e_2$. The same argument gives $e_2 \preceq e_3 \preceq \dots \preceq e_n \preceq e_1$, contradicting $e_1 \preceq e_n$. Thus $LD[Q]$ is acyclic, so each component is a tree directed toward a root of out-degree zero. Since $LD[Q]$ has only one vertex of out-degree zero, it has only one component.

11. Claim. Let $\mathcal{E}_{r,f}$ be the set of Euler tours of a directed graph G with first vertex r and first edge f , and let $\mathcal{A}_r$ be the set of spanning arborescences of G rooted at r . The map $LD: \mathcal{E}_{r,f}(G) \rightarrow \mathcal{A}_r$ given by $Q \mapsto LD(Q)$ is surjective.

Proof. Let $A_r \in \mathcal{A}_r$. The following procedure produces $Q \in \mathcal{E}_{r,f}$ with $LD(Q) = A_r$.

Procedure LD^{-1}

Input: a connected balanced digraph G ; a spanning arborescence A_r rooted at $r \in V(G)$.

Output: an Euler trail $Q = \langle r, f, v_1, e_1, \dots, e_{m-1}, r \rangle$ with $LD(Q) = A_r$.

Initialization.

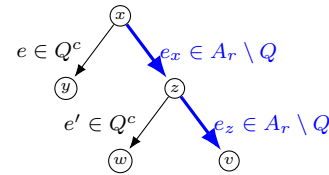
$$\begin{aligned} v_0 &:= r, \\ e_0 &:= f, \\ T_0 &:= \langle v_0, e_0, v_0 \rangle. \end{aligned}$$

Iteration. Given a trail $T_i = \langle v_0, e_0, \dots, e_{i-1}, v_i \rangle$, choose e_{i+1} as follows:

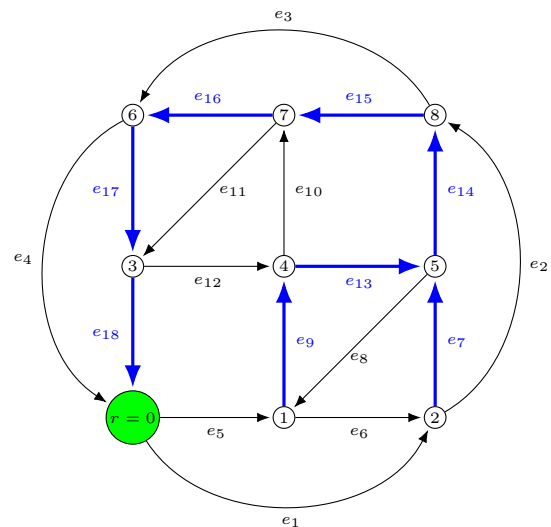
- e_{i+1} is any unused out-edge at v_i ;
- an edge of A_r is used only as a last resort (so that the last departure graph of the finished tour is A_r);
- stop if neither (a) nor (b) is possible.

Let Q be the trail at termination. Suppose Q is not an Euler circuit. Then some edge $e = (x, y)$ is unused. All edges incident with

r are used, so e is not incident with r . Since A_r is spanning, there is an edge $e_x = (x, z)$ of A_r leaving x . That edge cannot have been used, or else e would have been used.



Take e_x closest to the root in A_r . Since e_x is unused and G is balanced at z , some unused edge e' leaves z , and hence some unused edge of A_r leaves z , a contradiction. Thus Q is an Euler tour and $LD[Q] = A_r$.



An Euler tour in the preimage under LD of the (blue) arborescence A_r

12. Claim. Let $A_r \in \mathcal{A}_r$. The preimage of A_r under $LD: \mathcal{E}_{r,f}(G) \rightarrow \mathcal{A}_r$ contains precisely

$$|LD^{-1}(A_r)| = \prod_{v \in V \setminus \{r\}} (d_{\text{out}}(v) - 1)!.$$

tours in $\mathcal{E}_{r,f}$.

Proof. An Euler tour Q with last departure graph A_r induces, at each $v \neq r$, a permutation of the out-edges at v . The unique out-edge of A_r at v must be last, so there are $(d_{\text{out}}(v) - 1)!$ such permutations. The product over $V \setminus \{r\}$ follows. The root is excluded because a rotation of an Euler tour is the same tour.

13. Corollary. The number of Euler tours up to rotation is

$$|\mathcal{E}_{r,f}| = |\mathcal{A}_r| \prod_{v \in V \setminus \{r\}} (d_{\text{out}}(v) - 1)! \quad (1)$$

14. Corollary.

$|\mathcal{A}_r|$ is independent of the root r .

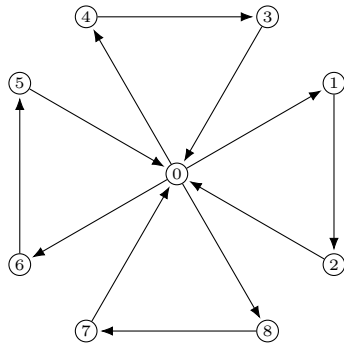
Proof. Rotation of a tour gives a bijection $\mathcal{E}_{r,f} \rightarrow \mathcal{E}_{r',f'}$, so $|\mathcal{E}_{r'}| = |\mathcal{E}_r|$. Equation (1) then yields $|\mathcal{A}_{r'}| = |\mathcal{A}_r|$.

15. An example is sometimes worth a thousand words. Consider the balanced connected digraph G in Figure (1a) which consists of four directed 3-cycles sharing the hub 0. By inspection, there is just a single in-arborescence rooted at each vertex so $|\mathcal{A}_r| = 1$ for any root r . Since G has 9

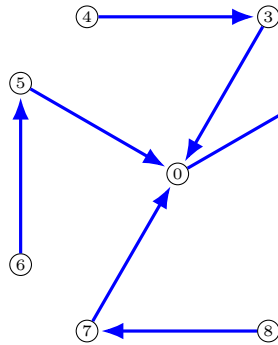
vertices, it follows that there are precisely 9 spanning arborescences (see Figure(3)). Figure (1b) draws the single arborescence A rooted at vertex 2. The elements of the preimage $LD^{-1}[A]$ are listed in Figure (1c) Hence, up to rotation, there are

$$|\mathcal{E}_{r,f}| = \prod_{v \in V \setminus \{r\}} (d_{\text{out}}(v) - 1)! = 6$$

Euler tours (each tour corresponding to one of the cyclic order of the four triangles.) The number of tours counting rotations (i.e. with a distinguished edge) is therefore $12 \times 6 = 72$. In particular, since there are 4 out edges, say f_1, f_2, f_3, f_4 at 0 there are $|\mathcal{E}_{f_1} \cup \mathcal{E}_{f_2} \cup \mathcal{E}_{f_3} \cup \mathcal{E}_{f_4}| = 4 \cdot 6 = 24$ tours which start at 0. Finally, Figure (2) shows the last departure graphs corresponding to a tour and a rotation of the tour.



(a) Balanced digraph G

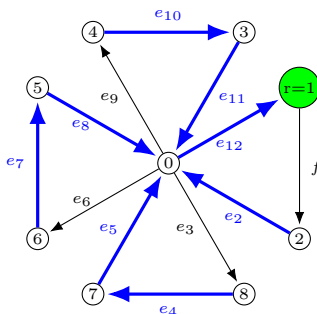


(b) $A \in \mathcal{A}_2$

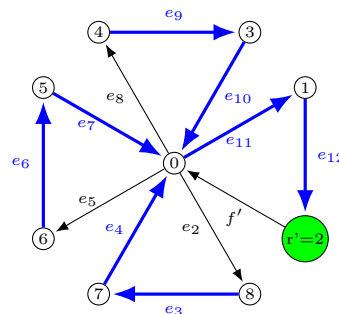
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(c) $LD^{-1}[A]$

Figure 1: The six Euler tours of the balanced digraph G which lie in the preimage of the spanning arborescence A (shown in blue).



(a) $A \in \mathcal{A}_1$



(b) $A' \in \mathcal{A}_2$

Figure 2: Distinct last departure graphs (in blue) determined by the rotation an Euler tour

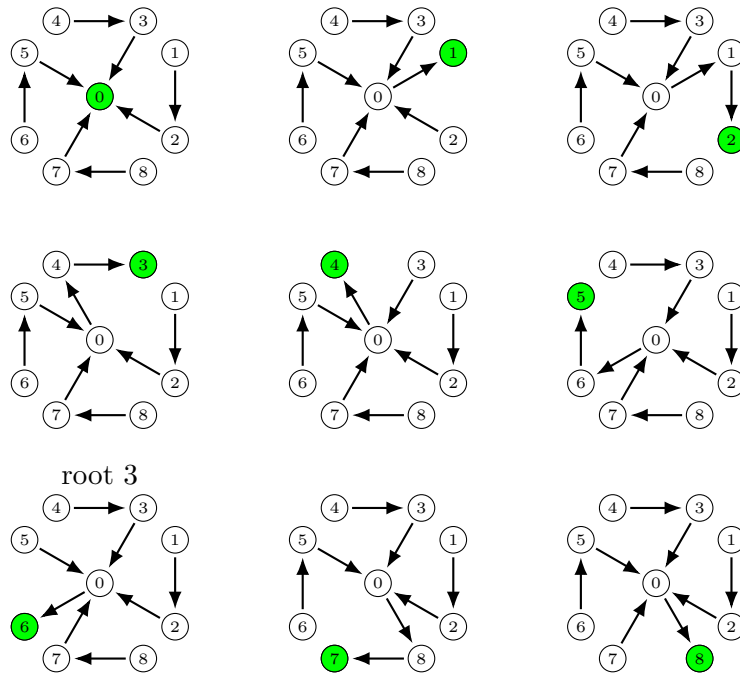


Figure 3: All spanning arborescences of G showing root vertices in green.