

Function Digraphs¹

1. A *digraph* is a pair $G = (V, E)$ where V is a set and $E \subseteq V \times V$.

2. Let $G = (V, E)$ be a digraph and $v \in V$. The *in-degree* and *out-degree* of v are defined by

$$d_{\text{in}}(v) = |\{w \in V : (w, v) \in E\}|,$$

$$d_{\text{out}}(v) = |\{w \in V : (v, w) \in E\}|,$$

respectively.

3. Remark. The in-degree and out-degree are also commonly denoted by $\deg^-(v)$ and $\deg^+(v)$.

4. A digraph is *balanced* if

$$d_{\text{in}}(x) = d_{\text{out}}(x), \quad \forall x \in V.$$

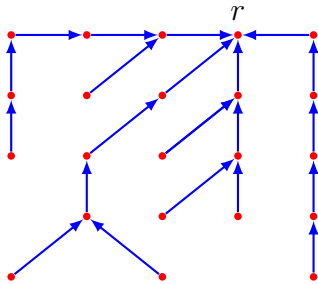
5. A digraph is called *connected* if it is connected as an undirected graph.

6. A digraph is a *directed cycle* if

(a) it is connected,

(b) $d_{\text{in}}(v) = d_{\text{out}}(v) = 1$ for all $v \in V$.

7. A tree with all edges directed to a root vertex of out-degree zero is called an *arborescence*.



8. A digraph $G = (V, E)$ is *oriented* if

$$(x, y) \in E \Leftrightarrow (y, x) \notin E$$

for all $x, y \in V$ (i.e., if it contains no directed cycle of length 2).

9. A function $f: V \rightarrow V$ determines a digraph $G_f = (V, E_f)$ where E_f is defined by

$$(x, y) \in E_f \Leftrightarrow y = f(x).$$

A digraph determined in this manner is called a *function digraph*.

10. Claim. Let F be a digraph with $d_{\text{out}}(v) \leq 1$ for all $v \in V(F)$. Then

(a) cycles in F are directed,

(b) paths in F are directed,

(c) cycles in F are chordless,

(d) different cycles of F are disjoint,

(e) no component of F can have more than one cycle.

Proof. (a) and (b) follow by induction (exercise). (c) and (d) follow from (a) by contradiction. To prove (e) it suffices to show that an arbitrary component C is either a tree or a tree plus one edge. Let T be a spanning tree of C so that $|E_T| \leq |E_C|$. Define a function $\phi: E_C \rightarrow V_C$ by $\phi(x, y) = x$. Since $d_{\text{out}} \leq 1$ it follows that ϕ is injective and so $|E_C| \leq |V_C|$. Since T is a spanning tree of C ,

$$|V_C| = |E_T| + 1 \leq |E_C| + 1,$$

and therefore

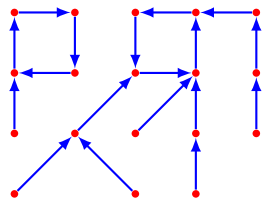
$$|V_C| - 1 \leq |E_C| \leq |V_C|.$$

If $|E_C| = |V_C| - 1$ then C is a tree (acyclic).

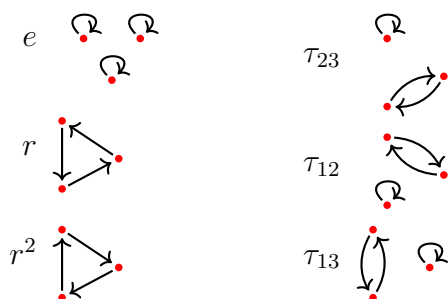
If $|E_C| = |V_C|$ then C has exactly one cycle.

11. Corollary. In a function digraph, $d_{\text{out}}(v) = 1$ for all $v \in V$ and so, in each component $|E_C| = |V_C|$. Thus each component contains a single cycle, which if contracted yields an arborescence with all paths directed toward the root.

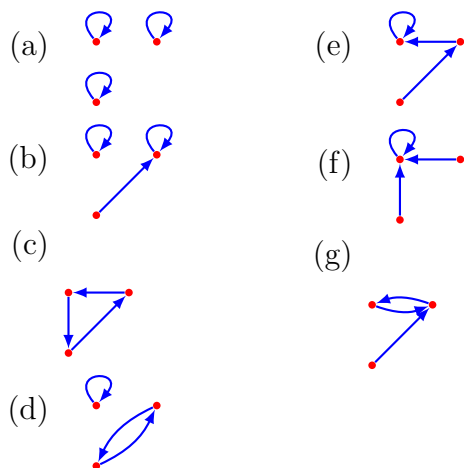
¹Partially based on a 1996 lecture by A. Kelmans at the University of Puerto Rico.



12. Special case. If $\sigma: V \rightarrow V$ is a permutation then $d_{\text{in}}(v) = 1$ for all $v \in V$. It follows that every component of the corresponding digraph of σ is a directed cycle. The following diagram shows all permutation digraphs on 3 vertices.



13. On a set of n vertices there are n^n function digraphs.
14. If $n = 3$ the 27 digraphs fall into 7 isomorphism classes with representatives:



15. If $n = 4$ the 256 digraphs fall into 20 isomorphism classes with representatives:

