

Simple Graphs¹

1. A *simple graph* G is an ordered pair (V, E) where V is a nonempty set of elements called vertices and $E \subseteq \binom{V}{2}$. The elements of E are called edges.
2. Let $G = (V, E)$ with $V = \{a, b, c\}$ and $E = \{\{a, b\}, \{a, c\}\}$. Then G is a simple graph. One drawing is



3. Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *equal*, written $G_1 = G_2$, if $V_1 = V_2$ and $E_1 = E_2$. Otherwise they are *different* ($G_1 \neq G_2$).
4. $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection $\alpha: V_1 \rightarrow V_2$ such that

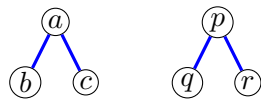
$$\{x, y\} \in E_1 \iff \{\alpha(x), \alpha(y)\} \in E_2. \quad (1)$$

5. Condition (??) is equivalent to

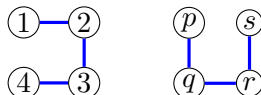
- (a) $\{x, y\} \in E_1 \implies \{\alpha(x), \alpha(y)\} \in E_2$,
- (b) $\{x, y\} \notin E_1 \implies \{\alpha(x), \alpha(y)\} \notin E_2$.

Thus an isomorphism maps edges to edges and non-edges to non-edges.

6. The bijection $\alpha: \{a, b, c\} \rightarrow \{p, q, r\}$ given by $\alpha(a) = p$, $\alpha(b) = q$, $\alpha(c) = r$ is an isomorphism of the graphs



7. Find an isomorphism between the graphs below.



Solution. Write G and H for the two graphs. Let $\alpha: V(G) \rightarrow V(H)$ be

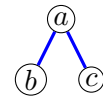
$$1 \mapsto s, \quad 2 \mapsto r, \quad 3 \mapsto q, \quad 4 \mapsto p.$$

Then

$$\begin{aligned} \{1, 2\} &\mapsto \{s, r\} \in E(H), \\ \{2, 3\} &\mapsto \{r, q\} \in E(H), \\ \{3, 4\} &\mapsto \{q, p\} \in E(H), \\ \{1, 3\} &\mapsto \{s, q\} \notin E(H), \\ \{1, 4\} &\mapsto \{s, p\} \notin E(H), \\ \{2, 4\} &\mapsto \{r, p\} \notin E(H). \end{aligned}$$

So α is a bijection sending edges to edges and non-edges to non-edges, hence an isomorphism.

8. A necessary condition for isomorphism is $|V_1| = |V_2|$ and $|E_1| = |E_2|$. In general, checking for isomorphism is a difficult problem. There is no short list of parameters that is also sufficient, and if $|V_1| = |V_2| = n$ there are $n!$ bijections $V_1 \rightarrow V_2$.
9. Let $G = (V, E)$ be a simple graph. A bijection $\alpha: V \rightarrow V$ is an *automorphism* of G if $\{x, y\} \in E \iff \{\alpha(x), \alpha(y)\} \in E$. An automorphism is an isomorphism from the graph to itself.
10. Let $G = (V, E)$ with $V = \{a, b, c\}$ and $E = \{\{a, b\}, \{a, c\}\}$.



The only automorphisms are id and the map α with $\alpha(a) = a$, $\alpha(b) = c$, $\alpha(c) = b$.

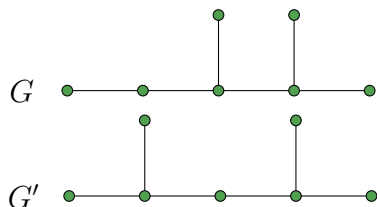


¹Notes based on Lectures by A. Kelmans at the University of Puerto Rico, 1996.

11. Let $\alpha: V \rightarrow V'$ be an isomorphism of simple graphs $G = (V, E)$ and $G' = (V', E')$. Then

- (a) $|V| = |V'|$,
- (b) $|E| = |E'|$,
- (c) $d_{G'}(\alpha(v)) = d_G(v)$ for all $v \in V$,
- (d) α maps cycles to cycles.

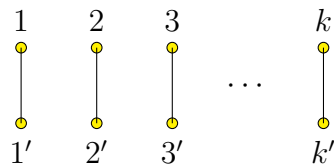
12. Consider the graphs G and G' drawn below.



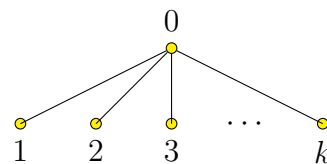
Then $G \not\cong G'$, since G' has a vertex adjacent to two vertices of degree 3 and G does not. Checking all $7!$ bijections is not efficient; in general the isomorphism problem is hard.

13. Isomorphism of simple graphs is an equivalence relation. Proof: exercise.
14. Exercise: Find the number of different simple graphs G with $V(G) \subseteq \{v_1, v_2, \dots, v_n\}$ that are isomorphic to each of the following.

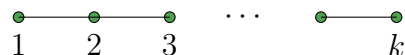
- (a) $k \leq n/2$.



- (b) $k \leq n - 1$.



- (c) $k \leq n$.



- (d) $k \leq n$.

