

The Euclidean Group

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1. Euler's rotation theorem.

Let $A \in O^+(3)$. Then there exists a non zero vector $x \in \mathbb{R}^3$ such that

$$Ax = x$$

Such a vector is called an axis of rotation.

Proof: We must show that the linear system

$$(A - I)x = 0$$

has a non trivial solution. This will occur if the determinant of the coefficient matrix $|A - I|$ is zero. Now $A \in O^+(3)$ implies ${}^tA = A^{-1}$ and so

$$\begin{aligned} |A - I| &= |{}^t(A - I)| \\ &= |{}^tA - I| \\ &= |A^{-1} - I| \\ &= |A^{-1}(I - A)| \\ &= |A^{-1}(-I)(A - I)| \\ &= |A^{-1}| |(-I)| |A - I| \\ &= -|A - I| \end{aligned}$$

Hence $|A - I| = 0$ as required. The

2. Claim

Let $F \in O^+(3)$. There exists an orthonormal basis (f_1, f_2, f_3) with respect to which the matrix A of F has the form

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proof: Let f_1, f_2, f_3 an orthonormal basis such that

$$F(f_3) = f_3$$

Expanding $F(f_1), F(f_2), F(f_3)$ in this basis:

$$\begin{aligned} F(f_1) &= \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \\ F(f_2) &= \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 \\ F(f_3) &= \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 f_3 \end{aligned}$$

Because $F(f_3) = f_3$ we must have $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = 1$. Since $F \in O(3)$ orthonormality of (f_1, f_2, f_3) induces orthonormality of $(F(f_1), F(f_2), F(f_3))$. Hence

$$\begin{aligned} \alpha_3 &= \langle F(f_1), f_3 \rangle \\ &= \langle F(f_1), F(f_3) \rangle \\ &= 0 \end{aligned}$$

Similarly $\beta_3 = 0$ and so

$$\begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $|F(f_1)| = 1$ we have

$$\alpha_1^2 + \alpha_2^2 = 1$$

and so there exists θ such that $\alpha_1 = \cos \theta$ and $\alpha_2 = \sin \theta$. From the orthonormality of $F(f_1), F(f_2)$ follows

$$\beta_1^2 + \beta_2^2 = 1$$

$$\alpha_1 \beta_1 + \alpha_2 \beta_2 = 0$$

Finally using $|A| = 1$ we deduce

$$A = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We note that $\cos \theta = \frac{\text{Tr} A - 1}{2}$ is independent of basis.

3. Special elements of $O(3)$

(a) Rotation:

$C_k(\theta)$ denotes the rotation fixing the line $0 + [k]$ with angle of rotation θ .

(b) Inversion:

The map $i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $x \mapsto -x$ is called *inversion*.

(c) Rotation-inversion:

$F \in O(3)$ of the form $i \circ C_k(\theta)$ is called a *rotation inversion*.

(d) Reflection:

A map of the form $\sigma_k = i \circ C_k(\pi)$ is called a *reflection*. It fixes the plane through the origin with normal k .

4. Remark:

Let $F \in O(3)$. Then F is either a rotation (orientation preserving) or a rotation inversion (orientation reversing).

Proof

If $F \notin O^+(3)$ then $F = i \circ (i \circ F)$.

5. Special Elements of $E(3)$:

- (a) $F \in E(3)$ is *rotation* if there exists a rotation $R \in O^+(3)$ and $T_a \in T(3)$ such that

$$F = T_a \circ R \circ T_{-a}$$

- (b) $F \in E(3)$ *reflection* if there exists a reflection $\sigma_k \in O(3)$ and $T_a \in T(3)$ such that

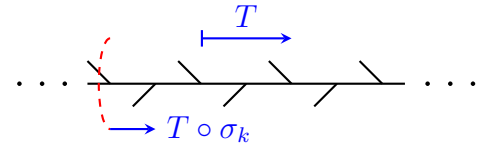
$$F = T_a \circ \sigma_k \circ T_{-a}$$

- (c) A *rotary reflection* (sometimes called a mirror rotation) is a rotation about some line ℓ followed by reflection in a plane perpendicular to ℓ .

(d) Screw Displacement

A *screw displacement* is a rotation followed by a translation along the rotation axis.

(e) Glide Reflection



A *glide reflection* is a reflection in a plane with normal σ_k followed by a translation perpendicular to k .

6. Example:

Let $k, p \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$: Then

$$F = T_p \circ C_k(\theta) \circ T_{-p}$$

is a rotation which fixes the line $p + [k]$.

7. Definition:

A *point group* is a group of isometries that keep at least one point fixed.

8. Examples of Point Groups

- (a) The *orthogonal group at p* :

$$O_p(3) = \{T_p \circ g \circ T_{-p} : g \in O(3)\}$$

- (b)

$$O_b^+(3) = \{T_b \circ g \circ T_{-b} : g \in O^+(3)\}$$

9. Claim An orientation preserving isometry of $E(3)$ is either:

- (a) The identity.
 (b) A translation.
 (c) A rotation about some line.
 (d) A screw rotation, that is a rotation about some line l followed by a translation parallel to l .

Proof:

If F is orientation preserving, it must have the form

$$F = T_a \circ C_k(\theta)$$

for some $a, k \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$. If $a = 0$ then F is a rotation about k . If $a \neq 0$ then either:

- (a) $a \perp k$
 (b) $a = a_{\parallel} + a_{\perp}$

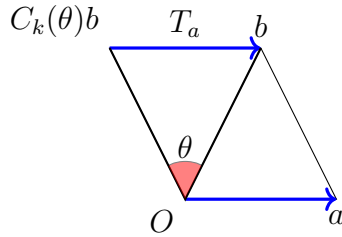
Case (a) If $a \perp k$ we claim that there exists $b \in \mathbb{R}^3$ such that

$$T_a \circ C_k(\theta) = T_b \circ C_k(\theta) \circ T_{-b}$$

i.e., such that

$$b - C_k(\theta)b = a$$

The existence of b follows from the fact that $b - C_k(\theta)b \perp k$ and $a \perp k$ (exercise: prove this fact). We conclude that F is a rotation fixing the line $b + [k]$



Case (b): If $a = a_{||} + a_{\perp}$ then

$$F = T_{a_{||}} \circ T_{a_{\perp}} \circ C_k(\theta)$$

By case (a),

$$T_{a_{\perp}} \circ C_k(\theta)$$

is a rotation fixing the line b and so F is a screw rotation.

10. Claim: An orientation reversing isometry of $E(3)$ is either:

- (a) A reflection in some plane π .
- (b) A glide reflection.
- (c) A rotatory reflection (mirror rotation), a rotation $C_k(\theta)$ about some axis k followed by reflection

Proof: As before, write $F = T \circ M$ where T is a translation and $M \in O(3)$. Since $|M| = -1$ it must have the form $M = i \circ C_k(\theta)$. Thus

$$\begin{aligned} F &= T_a \circ i \circ C_k(\theta) \\ &= T_a \circ i \circ C_k(\pi) C_k(\theta - \pi) \\ &= T_a \circ \sigma_k \circ C_k(\phi) \end{aligned}$$

where $\sigma_k = i \circ C_k(\pi)$ and $\phi = \pi - \theta$. If $a = 0$ we see that F is a rotary reflection.

If $a \perp k$ we claim that there exists $b \in \mathbb{R}^3$ such that

$$T_a \circ \sigma_k \circ C_k(\phi) = T_b \circ \sigma_k \circ C_k(\phi) \circ T_{-b}$$

i.e., such that

$$b - \sigma_k \circ C_k(\theta)b = a$$

The existence of b follows from the fact that $b - \sigma_k C_k(\theta)b \perp k$ and $a \perp k$. This shows that F is a rotary reflection fixing the line $b + [k]$. Finally, if $a = a_{||} + a_{\perp}$ then by the previous case F is a composition of a rotary reflection followed a translation in the direction parallel to the fixed line and so has no fixed point. It can be shown that an orientation reversing isometry with no fixed point must be a glide transformation.

11. Corollary: The set of fixed points of $F \in E(3)$ is either empty or an affine space. In $E(3)$ only the identity transformation leaves the whole space fixed. Reflections fix a plane and rotations fix a line.