# The Vector Product

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1. Claim.

   There exists exactly one bilinear functions $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ such that for all vectors $a, b \in \mathbb{R}^3$:

$$f(a,a) = 0 \tag{1}$$
$$f(a,b) \perp a \tag{2}$$
$$f(a,b) \perp b \tag{3}$$
$$\left.\begin{array}{l} |a| = |b| = 1 \\ \langle a, b\rangle = 0 \end{array}\right\} \implies |f(a,b)| = 1 \tag{4}$$
$$f(e_1, e_2) = e_3 \tag{5}$$

   Proof.

   Let $a, b \in \mathbb{R}^3$ and suppose that $f$ satisfies the conditions of the theorem.

   By (1) and bilinearity

$$\begin{aligned} 0 &= f(a+b, a+b) \\ &= f(a,a) + f(a,b) + f(b,a) + f(b,b) \\ &= f(a,b) + f(b,a) \end{aligned}$$

   Hence

$$f(a,b) = -f(b,a) \tag{6}$$

   i.e., $f$ is antisymmetric. If

$$a = \sum_{i=1}^{3} a_i e_i \text{ and } \quad b = \sum_{j=1}^{3} b_j e_j$$

   then by bilinearity,

$$\begin{aligned} f(a,b) &= f\left(\sum_{i=1}^{3} a_i e_i, \sum_{j=1}^{3} b_j e_j\right) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j f(e_i, e_j) \end{aligned}$$

   Using (1) and (6), this reduces to

$$\begin{aligned} f(a,b) \;=\; & (a_2b_3 - a_3b_2) f(e_2, e_3) \\ & + (a_3b_1 - a_1b_3) f(e_3, e_1) \\ & + (a_1b_2 - a_2b_1) f(e_1, e_2) \end{aligned}$$

   Properties (2),(3),(4) imply that

$$\begin{aligned} f(e_1, e_2) &= \pm e_3 \\ f(e_2, e_3) &= \pm e_1 \\ f(e_3, e_1) &= \pm e_2 \end{aligned}$$

   Specification of $f(e_1, e_2)$ by (5) determines $f(e_2, e_3)$ and $f(e_3, e_1)$ and hence $f$.

   In particular, if $f(e_1, e_2) = e_3$ then,

$$\begin{aligned} f(e_1, e_2 + e_3) &= f(e_1, e_2) + f(e_1, e_3) \\ &= e_3 \pm e_2 \end{aligned}$$

   Since $f(e_1, e_2 + e_3) \perp (e_2 + e_3)$ it must be that $f(e_1, e_3) = -e_2$. By antisymmetry, $f(e_3, e_1) = e_2$. A similar argument leads to $f(e_2, e_3) = e_1$. Thus, if $f$ satisfies properties (1) – (5), then necessarily

$$\begin{aligned} f(a,b) \;=\; & (a_2b_3 - a_3b_2) e_1 \\ & + (a_3b_1 - a_1b_3) e_2 \\ & + (a_1b_2 - a_2b_1) e_3 \end{aligned}$$

   It is left as an exercise to show the function $f$ given by this equation is bilinear and satisfies properties (1) – (5). We remark that (4) follows from Lagrange's identity (see below).

2. Definition.

   The unique bilinear function $f(a, b)$ satisfying properties (1) – (5) is called the *vector* or *cross* product of $a$ and $b$ and is denoted $a \times b$. Thus

$$\begin{aligned} a \times b \;=\; & (a_2b_3 - a_3b_2) e_1 \\ & + (a_3b_1 - a_1b_3) e_2 \\ & + (a_1b_2 - a_2b_1) e_3 \end{aligned}$$

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3. Lagrange Identity

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a \cdot b)^2.$$

Proof.
Computation reduces both sides to

$$a_2^2 b_3^2 + a_3^2 b_2^2 + a_3^2 b_1^2 + a_1^2 b_3^2 + a_1^2 b_2^2 + a_2^2 b_1^2 - 2a_2 a_3 b_2 b_3 - 2a_3 a_1 b_3 b_1 - 2a_1 a_2 b_1 b_2$$

Corollary. [Geometric Interpretation]
If $\theta \in [0, \pi]$ is the unoriented angle between $a$ and $b$ then

$$|a \times b| = |a||b| \sin\theta$$

4. The scalar triple product

$$\begin{aligned} \det[a, b, c] &= a \cdot (b \times c) \\ &= b \cdot (c \times a) \\ &= c \cdot (a \times b) \end{aligned}$$

Remark. The scalar triple product equals the signed volume of the parallelepiped determined by the three vectors. It is zero if the vectors are coplanar.

5. Vector Triple Product.

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

Proof. Exercise.
Hint: The maps $L, L' : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L(a) = a \times (b \times c)$$

$$L'(a) = (a \cdot c)b - (a \cdot b)c$$

are linear. It suffices to show that they are equal on the standard basis $e_1, e_2, e_3$.

6. Jacobi Identity

$$\begin{aligned} & a \times (b \times c) \\ + \; & b \times (c \times a) \\ + \; & c \times (a \times b) \\ = \; & 0 \end{aligned}$$

In Lie bracket notation:
$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

7. Derivative Formula

$$\frac{d}{dt}(a \times b) = \frac{da}{dt} \times b + a \times \frac{db}{dt}$$

8. $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c.$

Operator version:

$$\begin{aligned} \nabla \times (\nabla \times f) &= \nabla(\nabla \cdot f) - (\nabla \cdot \nabla) f \\ &= \nabla(\nabla \cdot f) - \nabla^2 f \end{aligned}$$

9. $(a \times b) \times (a \times c) = (a \cdot (b \times c))a$

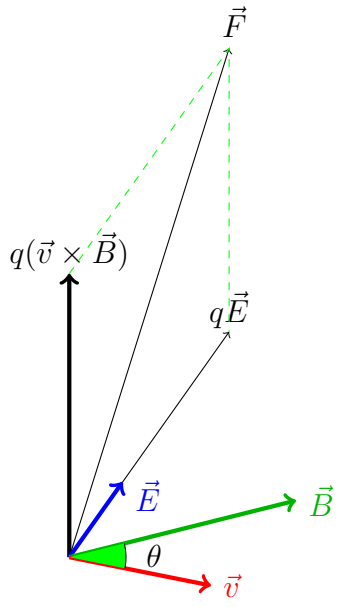
10. Binet-Cauchy identity

$(a \times b) \cdot (c \times d)$
$= (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c).$

11. The linear map $b \mapsto a \times b$ has matrix representation:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

12. The linear map $a \mapsto a \times b$ has matrix representation:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mapsto \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

13. Lorentz Force. $F = q(E + v \times B)$