Additional Properties of Subspaces

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1. Notation.

   Let V be a vector space. If W is a subspace of W we write $W \lhd V$.

2. Claim.

   Let V be a vector space.

   $$W, W' \lhd V \implies W \cap W' \lhd V.$$

   Proof.

   (a) Since W and W' are both vector spaces, $O_V \in W \cap W'$ and, therefore,

   $$W \cap W' \neq \emptyset.$$

   (b) Let $u, v \in W \cap W'$. Then $u, v \in W$. Since W is a subspace, $u + v \in W$. Similarly, $u + v \in W'$ and so

   $$u + v \in W \cap W'.$$

   (c) Finally, let c be a scalar and $u \in W \cap W'$. Since W is a subspace, $cu \in W$. Simlarly $cu \in W'$ and hence

   $$cu \in W \cap W'.$$

   Hence, by the subspace criterion

   $$W \cap W' \lhd V.$$

   Remark: This result is a special case of:

3. Claim.

   Let $\mathcal{W}$ be the set of subspaces of a vector space V.

   $$\mathcal{A} \subseteq \mathcal{W} \implies \bigcap \mathcal{A} \in \mathcal{W}$$

   I.e., $\mathcal{W}$ is a closure system on V.

4. Claim.

   Let V be a vector space. Then

   $$S \subseteq V \implies \langle S \rangle \lhd V.$$

5. Claim.

   Let V be a vector space and $S \subseteq V$.

   $$\langle S \rangle = \bigcap \{W \in \mathcal{W} : S \subseteq W\}.$$

   I.e., The span of S coincides with its *closure*.

6. Claim.

   Span is extensive, isotonic and idempotent. I.e., (respectively)

   (a) $A \subseteq \langle A \rangle$.

   (b) $A \subseteq B \implies \langle A \rangle \subseteq \langle B \rangle$.

   (c) $\langle \langle A \rangle \rangle = \langle A \rangle$.

7. Claim.

   $$W \lhd V \implies \langle W \rangle = W \qquad (1)$$

8. Claim.

   Let V be a vector space. Then

   $$W, W' \lhd V \implies W + W' \lhd V.$$

   where

   $$W + W' = \{w + w' : w \in W, \quad w' \in W'\}.$$

9. Definition.

   Let V be a vector space and $W, W' \lhd V$. If $W \cap W' = \{0_V\}$ then $W + W'$ is denoted $W \oplus W'$ and is called the *direct sum* of W and W'. In this case W and W' are said to be *supplementary* subspaces.

10. Claim.

    If $V = W_1 \oplus W_2$ and $v \in V$, there exist unique $w_1 \in W$ and $w_2 \in W_2$ such that

    $$v = w_1 + w_2.$$

    Proof.

    Suppose that

    $$\begin{aligned} v &= w_1 + w_2 \\ &= w_1' + w_2'. \end{aligned}$$

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then

$$W_1 \ni w_1 - w_1' = w_2 - w_2' \in W_2.$$

Hence

$$\begin{aligned} w_1 - w_1' &\in W_1 \cap W_2 \\ w_2 - w_2' &\in W_1 \cap W_2 \end{aligned}$$

But $W \cap W' = \{0_V\}$ and it follows that $w_1 = w_1'$ and $w_2 = w_2'$.

11. Claim.

Let V be a vector space and $W_1, W_2 \lhd V$. Then

$$\langle W_1 \cup W_2 \rangle = W_1 + W_2.$$

Proof.

Clearly

$$W_1 + W_2 \subseteq \langle W_1 \cup W_2, \rangle.$$

Let $x \in W_1 \cup W_2$. If $x \in W_1$ then

$$\begin{aligned} x &= w_1 + 0 \\ &\in W_1 + W_2 \end{aligned}$$

Similarly If $x \in W_1$ then

$$x \in W_1 + W_2.$$

It follows that

$$W_1 \cup W_2 \subseteq W_1 + W_2$$

and by (1):

$$\begin{aligned} \langle W_1 \cup W_2 \rangle &\subseteq \langle W_1 + W_2 \rangle \\ &= W_1 + W_2 \end{aligned}$$

12. Claim.

Let V be a finite dimensional vector spacen and $W_1, W_2 \lhd V$. Then

$$\begin{aligned} \dim(W_1 + W_2) = \dim W_1 + \dim W_2 \\ - \dim W_1 \cap W_2 \end{aligned} \quad (2)$$

Proof.

Let $\alpha \in \mathcal{B}(W_1 \cap W_2)$. Extend α to bases

$$M = \alpha \cup \beta \in \mathcal{B}(W_1)$$

$$N = \alpha \cup \gamma \in \mathcal{B}(W_2).$$

We Claim.

$$\alpha \cup \beta \cup \gamma \in \mathcal{B}(W_1 + W_2).$$

Clearly, $\alpha \cup \beta \cup \gamma$ is a spanning set.

Let

$$\begin{aligned} \alpha &= \{\alpha_1, \alpha_2, \ldots\} \\ \beta &= \{\beta_1, \beta_2, \ldots\} \\ \gamma &= \{\gamma_1, \gamma_2, \ldots\} \end{aligned}$$

If

$$\sum a_i \alpha_i + \sum b_j \beta_j + \sum c_k \gamma_k = 0$$

then

$$W_1 \ni \sum a_i \alpha_i + \sum b_j \beta_j = -\sum c_k \gamma_k \in W_2.$$

Hence

$$\sum a_i \alpha_i + \sum b_j \beta_j \in W_1 \cap W_2$$

and so

$$\sum a_i \alpha_i + \sum b_j \beta_j = \sum d_l \alpha_l.$$

i.e.,

$$\sum d_l \alpha_l - \sum a_i \alpha_i - \sum b_j \beta_j = 0.$$

Since $\alpha \cup \beta$ is an independent set, it must be that

$$b_j = 0, \forall j.$$

Similarly

$$c_k = 0, \forall k.$$

Hence

$$\sum a_i \alpha_i = 0.$$

But α is an independent set and so

$$a_i = 0, \forall i.$$

Hence $\alpha \cup \beta \cup \gamma$ is an independent set and it follows that

$$\alpha \cup \beta \cup \gamma \in \mathcal{B}(W_1 + W_2).$$

Therefore,

$$\begin{aligned} \dim(W_1 + W_2) &= |\alpha \cup \beta \cup \gamma| \\ &= |M \cup N| \\ &= |M| + |N| - |M \cap N| \end{aligned}$$

and so

$$\begin{aligned} \dim(W_1 + W_2) = \dim W_1 + \dim W_2 \\ - \dim W_1 \cap W_2. \end{aligned}$$

13. Example.

Let e_1, e_2, e_3 be the standard basis for $\mathbb{R}^3$. Consider the subspaces

$$W_1 = \langle e_1, e_2 \rangle$$
$$W_2 = \langle e_2, e_3 \rangle.$$

Then
$$W_1 \cap W_2 = \langle e_2 \rangle$$
has dimension 1. Hence, by (3)

$$\dim(W_1 + W_2) = 2 + 2 - 1 = 3.$$

14. Example.

Let e_1, e_2, e_3, e_4 be the standard basis for $\mathbb{R}^4$. Consider the subspaces

$$W_1 = \langle e_1, e_2 \rangle$$
$$W_2 = \langle e_3, e_4 \rangle.$$

Notice that the planes W_1 and W_2 intersect in a single point O and equation (3) yields
$$\dim(W_1 + W_2 = 4.$$

15. Let V be a finite dimensional vector space. and $W_1 \lhd V$ a subspace. Then there exists $W_2 \lhd V$ such that

$$W_1 \oplus W_2 = V.$$

Proof.

Let
$$\alpha = \alpha_1, \alpha_2, \ldots \alpha_k$$
be a basis for W_1. Extend α to a basis

$$\alpha_1, \ldots \alpha_k, \alpha_{k+1}, \ldots \alpha_n$$

for V. Let $W_2 = \langle \alpha_{k+1}, \ldots \alpha_n \rangle$. Clearly

$$V = W_1 + W_2.$$

We claim that

$$W_1 \cap W_2 = \{\emptyset\}.$$

Let $x \in W_1 \cap W_2$. Then, there exist constants $a_1, a_2, \ldots, a_n$ such that

$$x = \sum_{i=1}^{k} a_i \alpha_i = \sum_{i=k+1}^{n} a_i \alpha_i.$$

Hence,

$$\sum_{i=1}^{k} a_i \alpha_i + \sum_{i=k+1}^{n} (-a_i) \alpha_i = 0.$$

Since the α_i are independent, $a_i = 0$ for all i and so $x = O_V$.

Remark. In the special case, $V = \mathbb{R}^n$ if $W \lhd \mathbb{R}^n$ has an orthonormal basis

$$\alpha_1, \ldots \alpha_k$$

and
$$\alpha_1, \ldots \alpha_k, \alpha_{k+1}, \ldots \alpha_n$$
is an orthonormal extension to a basis for $\mathbb{R}^n$, then
$$\langle \alpha_{k+1}, \ldots \alpha_n \rangle$$
is a basis for the orthogonal subspace $W^\perp$.

16. Let V and W be finite dimensional vector spaces and $L : V \to W$ a linear map. Then there exists a *supplementary* subspace $P \lhd V$ such that

$$P \oplus \ker L = V.$$

Proof.

Let
$$\{w_1, w_2, \ldots w_k\}$$
be a basis for $\operatorname{Im} L$. Then there exist vectors $v_1, v_2 \ldots v_k \in V$ such that

$$L(v_i) = w_i, \quad 1 \leq i \leq k.$$

We Claim.

that the v_i comprise an independent set. Suppose, to the contrary, that there exist scalars $c_1, c_2, \ldots, c_k$, not all zero, such that

$$\sum_{i=1}^{k} c_i v_i = 0$$

then

$$\begin{aligned} L\left(\sum_{i=1}^{k} c_i v_i\right) &= \sum_{i=1}^{k} c_i L_i(v_i) \\ &= \sum_{i=1}^{k} c_i w_i \\ &= 0. \end{aligned}$$

Contradicting the independence of the w_i. Let

$$P = \langle v_1, v_2, \dots v_k \rangle.$$

We claim that

$$P \oplus \ker L = V.$$

If $v \in V$ then $L(v) \in \operatorname{Im} L$ and so there exist constants a_i such that

$$\begin{aligned} L(v) &= \sum a_i w_i \\ &= \sum a_i L v_i \\ &= L\left(\sum a_i v_i\right). \end{aligned}$$

Hence

$$L\left(v - \sum a_i v_i\right) = 0.$$

and so

$$\begin{aligned} v &\in \langle v_1, v_2, \dots, v_k \rangle + \ker L \\ &= P + L. \end{aligned}$$

It remains to show that

$$P \cap \ker L = \{0_V\}.$$

Let $x \in P \cap \ker L$. Then there exist constants d_i such that $x = \sum d_i v_i$ and

$$\begin{aligned} L(x) &= L\left(\sum d_i v_i\right) \\ &= \sum d_i L(v_i) \\ &= \sum d_i w_i \\ &= 0_V. \end{aligned}$$

By independence of the w_i we have that $d_i = 0$ for all i and so $x = 0$. Hence

$$P \oplus \ker L = V.$$

17. Corollary.

$$\dim(\ker L) + \dim(\operatorname{Im} L) = \dim V. \qquad (3)$$

18. Exercise.

Let $L : V \to W$ be a linear map. Show that the following are equivalent:

(a) L is injective.

(b) $\ker L = \{O_V\}$.

19. Claim.

Let $L : V \to W$ be a linear map. Show that

(a) If L is injective, then

$$\dim(V) \leq \dim W.$$

(b) If L is surjective then

$$\dim(V) \geq \dim W.$$

Proof.

Follows immediately from (3) by noting that if L is injective, then

$$\dim(\ker L) = 0$$

whereas if L is surjective

$$\dim(\operatorname{Im} L) = \dim(W).$$

Exercises

1. Which of the following sets are subspaces of R^3? Justify your answer.

   (a) $\{(a, b, c) | 2a - b + c = 0\}$

   (b) $\{(0, 0, 0)\}$

   (c) $\{(x, y, z) | z + xy = 0\}$

   (d) $\{(x, y, z | x + y = 0\}$

   (e) $\{(a, b) \in R^2 : b = a^2\}$

2. Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be vectors. Suppose $\vec{u}$ lies in the span of $\vec{v}$, and $\vec{w}$. Show that $\{\vec{v},\ \vec{w}\}$ spans the same subspace as $\{\vec{u}, \vec{v}, \vec{w}, \}$.

3. Let S be a subspace of R^n. Define

   $$S^\perp = \{x \in R^n : \langle x, s \rangle = 0 \quad \forall s \in S\}$$

   Show that $S^\perp$ is a subspace of R^n.

4. Let W and W' be subspaces of a vector space V. Show that $W \bigcap W'$ is a subspace of V.

5. Show that the intersection of any family of subspaces of a vector space V is a subspace.

6. Let W be the set of solutions of the differential equation $x'' + x = 0$.

   (a) Show that the set $\{\sin, \cos\}$ is a linearly independent subset of the vector space V of functions with derivatives of all orders in $\mathbb{R}$.

   (b) Let

   $$f(t) = \sin t$$
   $$g(t) = \cos(t + \pi/2)$$

   Show that $\{f, g\}$ is a linearly dependent subset of W.

7. Let V be a finite dimensional vector space and W_1, W_2 subspaces of V. Show that

   $$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim W_1 \bigcap W_2$$

8. Let V be a vector space and let $\mathcal{W}$ be the set of all subspaces of V. Let $A \subseteq V$. The *closure* $\bar{A}$ of A is the set

   $$\bar{A} = \bigcap \{W \in \mathcal{W} : A \subseteq W\}$$

   Show that $\bar{A}$ is a subspace. Show also that:

   (a) $A \subseteq B \implies \bar{A} \subseteq \bar{B}$ (Isotonicity)

   (b) $A \subseteq \bar{A}$ (Extensivity)

   (c) $\bar{\bar{A}} = \bar{A}$ (Idempotence)

9. Let V be a vector space and $A \subseteq V$. Show that

   $$\bar{A} = \langle A \rangle$$

10. Exercise.

    Let $L : V \to W$ be a linear map. Show that if $\dim(V) = \dim(W)$, then L is injective if and only if it is surjective.

11. Exercise.

    Let $L : V \to W$ be a linear map. If $v, v' \in V$ define

    $$v \sim v' \iff L(v) = L(v').$$

    (a) Show that $\sim$ is an equivalence relation.

    (b) Show that the equivalence classes of $\sim$ are the distinct cosets of the kernel of L .

    (Recall that if $x \in V$ the *coset* determined by x is the set $\{x + k : k \in \ker L\}$.)