Matrix Multiplication – Brief Review

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1. Let K be a field and let

$$A \in K^{m\times n}$$

be an $m \times n$ matrix over K. Denote by $A(i,j)$ the element of A located in row i and column j. The matrix A determines *row matrices* $A_1, \ldots, A_m \in K^{1\times n}$ and *column matrices* $A^1 \ldots, A^n \in K^{m\times 1}$ defined (respectively) by:

$$\begin{aligned} A_i(1,j) &= A(i,j) && (1)\\ A^j(i,1) &= A(i,j) && (2) \end{aligned}$$

where $i = 1, \ldots, m; \quad j = 1, \ldots n.$

2. Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \pi \end{bmatrix}$ then

$A_2 = \begin{bmatrix} 4 & 5 & \pi \end{bmatrix}$ and $A^3 = \begin{bmatrix} 3 \\ \pi \end{bmatrix}$. Notice that $A_2(1,3) = A(2,3) = \pi$ and also $A^3(2,1) = A(2,3) = \pi$

3. Definition:

Let $x = (x_k)$ be a column vector in K^n and let $A \in K^{m\times n}$ an $m \times n$ matrix. The *product of A with x* is given by:

$$Ax = \sum_{j=1}^{n} x_j A^j$$

This makes Ax a linear combination of the columns of A. The ith element of the column Ax is

$$(Ax)(i,1) = \sum_{k=1}^{n} A(i,k)x_k$$

4. Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

5. Definition:

Let $A \in K^{m\times n}$ and $B \in K^{n\times p}$. The *product* of A and B is the matrix $AB \in K^{m\times p}$ whose columns are determined by:

$$(AB)^j = AB^j, \quad j = 1, \ldots p$$

6. Let A $= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

The second column of AB is:

$$\begin{aligned} (AB)^2 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} \\ &= 6 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 22 \\ 50 \end{bmatrix} \end{aligned}$$

7. Claim:

If $A \in K^{m\times n}$ and $B \in K^{n\times p}$

$$AB(i,j) = \sum_{k=1}^{n} A(i,k)B(k,j) \qquad (3)$$

Proof:

$$\begin{aligned} (AB)^j &= AB^j \\ &= \sum_{k=1}^{n} B^j(k,1)A^k \end{aligned}$$

Equating components gives:

$$\begin{aligned} (AB)^j(i,1) &= \sum_{k=1}^{n} B^j(k,1)A^k(i,1) \\ &= \sum_{k=1}^{n} A^k(i,1)B^j(k,1) \end{aligned}$$

Finally by (2)

$$AB(i,j) = \sum_{k=1}^{n} A(i,k)B(k,j)$$

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8. Special cases of (3):

   (a) If $x = (x_k)$ is a column vector in K^n. Regarded as a matrix $x_k = x(k,1)$. If $A \in K^{m\times n}$ is an $m \times n$ matrix then

$$\begin{aligned} Ax(i,1) &= \sum_{k=1}^{n} A(i,k)x(k,1) \\ &= \sum_{k=1}^{n} x_k A^k(i,1) \end{aligned}$$

   and so, in agreement with the definition,

$$Ax = \sum_{k=1}^{n} x_k A^k$$

   (b) If $x = (x^k)$ is a row vector in K^m. As a matrix $x^k = x(1,k)$. If $A \in K^{m\times n}$ is an $m \times n$ matrix then

$$\begin{aligned} xA(1,j) &= \sum_{k=1}^{m} x(1,k)A(k,j) \\ &= \sum_{k=1}^{m} x^k A^k(1,j) \end{aligned}$$

   and so

$$xA = \sum_{k=1}^{m} x^k A_k$$

   is a linear combination of the rows of A.

9. Example:

$$(5,6)\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 5(1,2) + 6(3,4)$$

10. Exercise.
    If $A \in K^{m\times n}$ and $B \in K^{n\times p}$ show

$$(AB)_i = A_i B$$

11. Exercise.
    Let $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ and $B \in K^{2\times 2}$. Show:

    (a) $(AB)_1 = \alpha B_1 + \beta B_2$

    (b) $(AB)_2 = \gamma B_1 + \delta B_2$

    Find the values of α, β, γ and δ that carry out the elementary row operation $B_2 \mapsto B_2 - \lambda B_1$

12. Claim:
    If $A \in K^{m\times n}$ and $B \in K^{n\times p}$

$$AB = \sum_{i=1}^{n} A^i B_i$$

    Proof.

$$\begin{aligned} \sum_{k=1}^{n} A^k B_k(i,j) &= \sum_{k=1}^{n}\sum_{l=1}^{1} A^k(i,l)B_k(l,j) \\ &= \sum_{k=1}^{n} A^k(i,1)B_k(1,j) \\ &= \sum_{k=1}^{n} A(i,k)B_k(k,j) \\ &= (AB)(i,j) \end{aligned}$$

13. Example

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 6 & 7 \end{bmatrix} \\ &\quad + \begin{bmatrix} 2 \\ 4 \end{bmatrix}\begin{bmatrix} 8 & 9 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 & 7 \\ 15 & 18 & 21 \end{bmatrix} + \begin{bmatrix} 16 & 18 & 20 \\ 32 & 36 & 40 \end{bmatrix} \end{aligned}$$

    (a sum of rank 1 matrices.)

14. Definition:
    Let $A \in K^{m\times n}$. The *tranpose* of A is the $n \times m$ matrix ${}^tA$ defined by

$${}^tA(i,j) = A(j,i)$$

    where $1 \le i \le n$, and $1 \le j \le m$.

15. Example:

    Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \pi \end{bmatrix}$ then ${}^tA = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \pi \end{bmatrix}$.

16. Claim:

$$ {}^t(A^k) = ({}^tA)_k $$

Proof:

$$ \begin{aligned} {}^t(A^k)(1,j) &= A^k(j,1) \\ &= A(j,k) \\ &= {}^tA(k,j) \\ &= ({}^tA)_k(1,j) \end{aligned} $$

17. Exercise:

Show that:

(a) ${}^{tt}A = A$

(b) ${}^t(A_k) = ({}^tA)^k$

(c) ${}^t(AB) = {}^tB\ {}^tA$

(d) $A_i = {}^t(({}^tA)^i)$

(e) $A^i = {}^t(({}^tA)_i)$

(f) $(AB)_i = A_iB$

18. Remark:

Let x, y are row vectors in K^n. Regarding x and y as $1 \times n$ matrices, it is seen that the inner product $\langle x, y\rangle$ coincides with the matrix product $x({}^ty)$. It follows that if $A \in K^{m\times n}$ and $B \in K^{n\times p}$ then

$$ (AB)(i,j) = \langle A_i, {}^t(B^j)\rangle $$
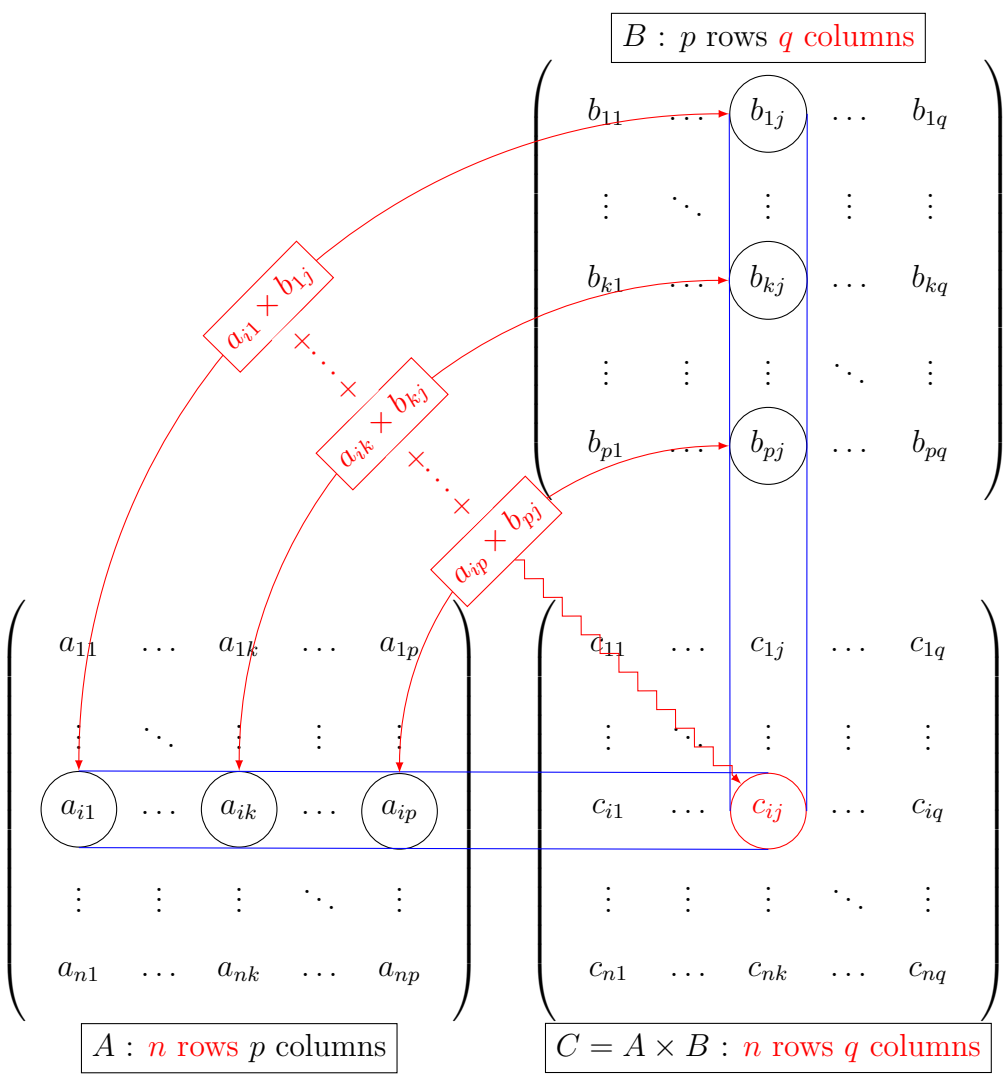
Illustration of formula $AB(i,j) = \sum_{k=1}^{n} A(i,k)B(k,j)$ –by Alain Matthes