The Inverse of a Matrix.

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1. Formulae for Matrix Multiplication

   If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times p}$ then

   (a) $AB(i,j) = \sum_{k=1}^{n} A(i,k)B(k,j)$

   (b) $(AB)^i = AB^i$ – a linear combination of the columns of A

   (c) $(AB)_i = A_iB$ – a linear combination of the rows of B

   (d) $AB = \sum_{i=1}^{n} A^iB_i$

2. A Column × a Row.

$$\begin{pmatrix}2\\3\\4\end{pmatrix}\begin{pmatrix}1 & 6\end{pmatrix} = \begin{pmatrix}2 & 12\\3 & 18\\4 & 24\end{pmatrix}$$

   Notice that all rows are multiples of the vector $(1,6)$ and both columns are multiples of ${}^t(2,3,4)$.

3. Example of (d)

$$\begin{pmatrix}2 & 7\\3 & 8\\4 & 9\end{pmatrix}\begin{pmatrix}1 & 6\\0 & 0\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix}\begin{pmatrix}1 & 6\end{pmatrix} + \begin{pmatrix}7\\8\\9\end{pmatrix}\begin{pmatrix}0 & 0\end{pmatrix}$$

   The result is a sum of rank matrices of rank at most 1.

4. Inverses.

   A square matrix A is *invertible* or *non singular* if there exists a square matrix B such that $BA = I = BA$. The inverse matrix B if it exists is unique and and is denoted by A^{-1} so that $A^{-1}A = I = AA^{-1}$.

5. Singular case (no inverse). $\begin{pmatrix}1 & 3\\2 & 6\end{pmatrix}$ Has no inverse. Since $(AB)^i$ is a linear combinations of the columns of A and all columns of A are multiples of the first column we cannot produce column 1 of the identity. $\begin{pmatrix}1\\0\end{pmatrix}$ by multiplication. Note for later: it is not an accident that the determinant is zero.

6. Claim.

   A square matrix has no inverse if there exists a non zero vector x such that $Ax = \vec{0}$ (i.e., if some non trivial linear combination of the columns of A gives the zero vector).

   Proof.

   If A had an inverse B, then

$$\begin{aligned} Ax = 0 &\Rightarrow BAx = \vec{0}\\ &\Rightarrow Ix = \vec{0}\\ &\Rightarrow x = \vec{0}\end{aligned}$$

7. Computation of A inverse.

$$\begin{pmatrix}1 & 3\\2 & 7\end{pmatrix}\begin{pmatrix}a & c\\b & d\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

   Must solve two systems

$$\begin{pmatrix}1 & 3\\2 & 7\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 3\\2 & 7\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

   Gauss-Jordan Idea –Solve both at once! Put both columns in augmented matrix and do elimination as before. Inverse will show up on right.

$$\left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\2 & 7 & 0 & 1\end{array}\right] \sim \left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\0 & 1 & -2 & 1\end{array}\right] \sim \left[\begin{array}{cc|cc}1 & 0 & 7 & -3\\0 & 1 & -1 & 2\end{array}\right]$$

   $\begin{bmatrix}7 & -3\\-1 & 2\end{bmatrix}$ is the required inverse. This method works because if E is a product of elementary matrices with $EA = I$ then $E = A^{-1}$ and so $EI = E = A^{-1}$.

[1] http://pennance.us Based on MIT Lecture 3 by Gilbert Strang