Finding the Nullspace of a Matrix

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1. To find the null space of a matrix A, recall that if $A \sim H$ where H has Hemite normal form then
$$N(A) = N(H).$$
The following example illustrates the computation of $N(H)$.

2. Example:
Suppose
$$H = \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$N(H)$ is the set of solutions of the system
$$\begin{aligned} x_1 + 2x_2 - 2x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$
Let F, P be the sets of free and pivot variables.
$$\begin{aligned} P &= \{x_1, x_3\} \\ F &= \{x_2, x_4\} \end{aligned}$$
Notice that any assignment of values to the free variables yields a solution of the system. In particular, the assignments
$$\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
yield linearly independent solutions:
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$
Since $\dim N(A) = 2$ the two solutions form a basis.

3. Remark: If the variables are relabelled so that the pivot variables are in the initial columns.
$$H = \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ \hline \multicolumn{2}{c|}{O} & \multicolumn{2}{c}{O} \end{array}\right]$$
which has the form
$$\left[\begin{array}{c|c} I & F \\ \hline O & O \end{array}\right]$$
Then
$$H \left[\begin{array}{c} -F \\ \hline I \end{array}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
Thus, up to a relabelling of the variables the columns of $\left[\begin{array}{c} -F \\ \hline I \end{array}\right]$ yield a basis for $N(A)$. In this case
$$\left[\begin{array}{c} -F \\ \hline I \end{array}\right] = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. Example.
Let $\alpha = (1, 2, 3, 4) \in \mathbb{R}^4$. Find a basis for the subspace $\alpha^\perp$.
Solution. Let $v = (a, b, c, d) \in \mathbb{R}^4$. Then,
$$\begin{aligned} v \in \alpha^\perp &\Leftrightarrow a + 2b + 3c + 4d = 0 \\ &\Leftrightarrow v^T \in N(H) \end{aligned}$$
where $H = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$. Notice that H is in Hermite normal form with free

[1] http://pennance.us

variables $F = \{x_2, x_3, x_4\}$. Every assignment of values to the free variables yields a solution. In particular, the choices:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}; \quad \begin{bmatrix}0\\1\\0\end{bmatrix}; \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

yield three linear independent solutions:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}; \quad \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}; \quad \begin{bmatrix}-4\\0\\0\\1\end{bmatrix}$$

These vectors comprise a basis for $N(A)$ and hence $\alpha^\perp$.

5. Example. Find a basis for the subspace

$$W = \left\{ \begin{bmatrix}a & b\\c & d\end{bmatrix} : \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \right\}$$

Solution.
Notice,

$$v \in W \Leftrightarrow \begin{cases} a + 3b = 0 \\ c + 3d = 0 \end{cases}$$
$$\Leftrightarrow (a, b, c, d)^T \in N(A)$$

where $A = \begin{bmatrix}1 & 3 & 0 & 0\\0 & 0 & 1 & 3\end{bmatrix}$. Notice, A is in Hermite normal form. Since the variables x_2 and x_4 are free the assignements:

$$\begin{bmatrix}x_2\\x_4\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}; \quad \begin{bmatrix}x_2\\x_4\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

yield independent solutions:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-3\\1\\0\\0\end{bmatrix}; \quad \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\-3\\1\end{bmatrix}$$

comprising a basis for $N(A)$. The corresponding basis for W is

$$\left\{ \begin{bmatrix}-3 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\-3 & 1\end{bmatrix} \right\}$$

6. Example. Find a basis for the subspace:

$$W = \{p \in \mathcal{P}_3 : p(1) = 0;\ p(2) = 0\}$$

Solution.
Let $v = a + bx + cx^2 + dx^3 \in \mathcal{P}_3$.

Notice,

$$v \in W \Leftrightarrow \begin{cases} a + b + c + d = 0 \\ a + 2b + 4c + 8d = 0 \end{cases}$$
$$\Leftrightarrow (a, b, c, d)^T \in N(A)$$

where

$$A = \begin{pmatrix}1 & 1 & 1 & 1\\1 & 2 & 4 & 8\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & 3 & 7\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & 0 & -2 & -6\\0 & 1 & 3 & 7\end{pmatrix}$$

Since the variables x_3 and x_4 are free, the assignements:

$$\begin{bmatrix}x_3\\x_4\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}; \quad \begin{bmatrix}x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

yield the basis

$$\left\{ \begin{bmatrix}2\\-3\\1\\0\end{bmatrix}, \begin{bmatrix}6\\-7\\0\\1\end{bmatrix} \right\}$$

for $N(A)$. It is easy to verify that the corresponding polynomials

$$x^2 - 3x + 2$$
$$x^3 - 7x + 6$$

form a basis for W.