Convex Sets

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1. A subset S of an affine space E is called *convex* if

$$\forall x, y \in S, \quad \overline{xy} \subseteq S$$

2. Definition.

   Let E be an affine space and $x_1, x_2, \dots x_n \in E$. An affine combination $\sum \lambda_i x_1$ with $\sum \lambda_i = 1$ is called a *mass distribution* if $\lambda_i \geq 0$, $1 \leq i \leq n$..

3. Claim.

   Let $S = \{x_1, x_2, \dots x_n\}$ be a finite set of points in an affine space. Then the set $H(S)$ of all mass distributions of S is convex.

   Proof.

   Let $p, q \in H(S)$. Then there exist constants $\lambda_i, \lambda_i' \quad 1 \leq i \leq n$

$$p = \sum \lambda_i x_i, \quad \sum \lambda_i = 1, \quad \lambda_i \geq 0$$
$$q = \sum \lambda_i' x_i, \quad \sum \lambda_i' = 1, \quad \lambda_i' \geq 0$$

   If $0 < t < 1$ then

$$\begin{aligned}(1-t)p + tq &= \sum [(1-t)\lambda_i + t\lambda_i']x_i \\ &= \sum \mu_i x_i\end{aligned}$$

   where $\mu_i = (1-t)\lambda_i + t\lambda_i'$. Since $\sum \mu_i = 1$ and $\mu_i \geq 0, \quad 1 \leq i \leq n$,

$$(1-t)p + tq \in H$$

   Hence H is convex.

4. $H(S)$ is called the *convex hull* of S.

5. Examples

   (a) Let $x_1, x_2 \in \mathbb{R}^2$, then $H(x_1, x_2)$ is the segment

$$\lambda_1 x_1 + \lambda_2 x_2$$

   where $\lambda_1 + \lambda_2 = 1, \quad \lambda_i \geq 0$

   (b) Let $x_1, x_2, x_3 \in \mathbb{R}^2$ be non colinear, then $H(x_1, x_2, x_3)$ is a triangle.

   (c) Let $o = (0, 0)$ and let $x_1, x_2 \in \mathbb{R}^2$. Then $H(o, x_1, x_2, x_1 + x_2)$ is a paralellogram.

6. Claim.

   Let A be any convex set containing $S = \{x_1, x_2, \dots x_n\}$. Then $H(S) \subseteq A$.

   Proof.

   (By induction on $|S|$). If $S = \{x_1\}$ then $H(S) = S \subseteq A$ and the result is true. Let $n \in \mathbb{N}$ and S a set with n elements. Let $x \in S$. Then there exist $t_1, t_2, t_n \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n-1} t_i x_i + t_n x_n$$

   If $t_n = 1$ then $x = x_n \in A$. If $t_n \neq 1$ then

$$\begin{aligned}x &= (1-t_n) \sum_{i=1}^{n-1} \left(\frac{t_i}{1-t_n} \right) x_i + t_n x_n \\ &= (1-t_n)y + t_n x_n\end{aligned}$$

   where

$$y = \sum_{i=1}^{n-1} \frac{t_i}{1-t_n} x_i$$

   Since

$$\sum_{i=1}^{n-1} \frac{t_i}{1-t_n} = 1,$$

   it follows that $y \in H(x_1, x_2, \dots, x_{n-1})$ But A is a convex set containing $x_1, x_2, \dots x_n$ so by the induction hypothesis

$$H(x_1, x_2, \dots, x_{n-1}) \subseteq A,$$

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in particular $y \in A$. By convexivity of A it follows that $x \in A$ and so

$$H(x_1, x_2, \ldots, x_n) \subseteq A$$

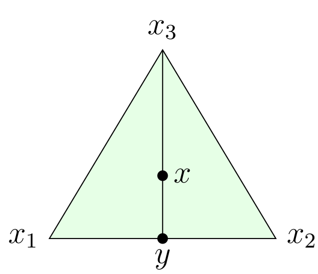
which was to be proven.

7. Illustration of the induction step of the proof for $n = 3$

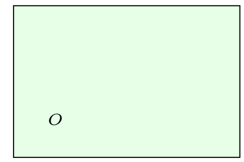
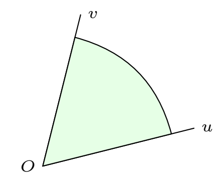
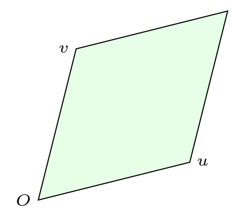
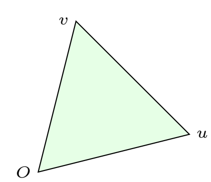
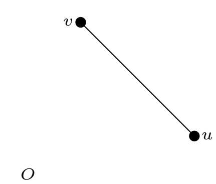
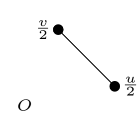
$$\begin{aligned} x &= \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \\ &= \frac{2}{3}(\frac{1}{2}x_1 + \frac{1}{2}x_2) + \frac{1}{3}x_3 \\ &= \frac{2}{3}y + \frac{1}{3}x_3 \end{aligned}$$

where

$$y = \frac{1}{2}x_1 + \frac{1}{2}x_2 \in H(x_1, x_2)$$



8. $\{t_1 u + t_2 v : t_1, t_2 \in \mathbb{R} + \text{Constraint}\}$

Constraint	Set	Drawing
$t_1, t_2 \in \mathbb{R}$	Plane	
$0 \le t_1, t_2$	Angle	
$0 \le t_1, t_2 \le 1$	Parallelogram	
$\begin{cases} 0 \le t_i \le 1 \\ t_1 + t_2 \le 1 \end{cases}$	Triangle	
$\begin{cases} 0 \le t_i \le 1 \\ t_1 + t_2 = 1 \end{cases}$	Segment	
$\begin{cases} 0 \le t_i \le 1 \\ t_1 + t_2 = \frac{1}{2} \end{cases}$	Segment	
$t_1 + t_2 = 1$	Line