Remarks on Strang's Video Lecture 4: LU Factorization

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1. A square matrix is called
   (a) *lower triangular* if all the entries above the main diagonal are zero.
   (b) *upper triangular* if all the entries below the main diagonal are zero.
   (c) *unit triangular* if it is triangular and all diagonal entries are 1.
   (d) *atomic triangular* or *Gaussian* if it is unit triangular and all off-diagonal elements are zero, except for the entries in a single column.

2. An atomic triangular matrix (a_{ij}) has the form:

$$\begin{bmatrix} 1 & & & & & & & \\ 0 & \ddots & & & \text{\Large 0} & & & \\ 0 & \ddots & 1 & & & & & \\ 0 & \ddots & 0 & 1 & & & & \\ & & 0 & a_{i+1,i} & 1 & & & \\ \vdots & & 0 & a_{i+2,i} & 0 & \ddots & & \\ & & \vdots & \vdots & \vdots & \ddots & 1 & \\ 0 & \dots & 0 & a_{n,i} & 0 & \dots & 0 & 1 \end{bmatrix}$$

3. LU Decomposition.
   Given a matrix A we seek a factorization:
$$A = LU$$
   where L is lower triangular matrix U upper triangular. If $n = 2$ this looks like:
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

4. Example.
   Let $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$. Since A is invertible, there exists a matrix E such that $EA = U$ where U has upper triangular form. In this case E, found by Gaussian elimination is an atomic lower triangular matrix,
$$E = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$
   and
$$\underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{U}$$
   Let
$$\begin{aligned} L &= E^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \end{aligned}$$
   Then
$$\underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{U}$$
   The right hand side can be rewritten:
$$\underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}_{U'}$$
   in which the upper triangular matrix U' is also a unit triangular and the matrix D is a diagonal matrix of pivots.

5. Exercise: If E is atomic lower triangular, then E^{-1} is obtained by reversing the signs of all off diagonal elements. For example:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

[1] http://pennance.us

6. If $A = (a_{ij})$ is a 3×3 matrix then LU decomposition decompostion takes the form:
$$A = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

7. If A can be row reduced without row permutations then there exist atomic lower triangular matrices E_{21}, E_{31}, E_{32} such that
$$E_{32}E_{31}E_{21}A = U$$
with U is upper triangular. In this case $A = LU$ where
$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$$
is a unit lower triangular.

8. Example (in which $E_{31} = I$)
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}}_{E_{32}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}}_{L}$$

9. Whenever the Gaussian elimination can be carried through without row permutations, the matrix L will be unit lower triangular and the elements below the diagonal will be just the multipliers used in the row operations.

10. In the case when an initial row permutation P is required (to avoid zeros in the pivot positions), the decomposition becomes $PA = LU$. This is called LU factorization with *partial pivoting*. It can be shown that all square matrices can be factorized in this form.

11. Complexity in the $n \times n$ case.
Using the relation
$$1^2 + 2^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1)$$
it can be shown that the number of operations required to factor A into LU is $\sim n^3$ for large n (see video).

12. Efficient algorithm to solve $Ax = b$ where A is $n \times n$.
    (a) Find (by Gaussian elimination) P, L, U such that
    $$PA = LU$$
    (b) Solve the equation $Lc = Pb$ for c by forward substitution.
    (c) Solve the equation $Ux = c$ for x by back substitution.

The total cost is approximately $\frac{1}{3}n^3$

13. Once L, U, P are known, the system $Ax = b$ can be solved for each b at relatively small cost ($\sim n^2$).

14. Example: Solve by LU algorithm
$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= 5 \end{aligned}$$
Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$
and $b = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix}^T$ Gaussian elimination yields (exercise):
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$
Let $c = Ux$. Then $Lc = b$ is the system
$$\begin{aligned} c_1 &= 1 \\ c_1 + c_2 &= 3 \\ c_1 + c_2 + c_3 &= 5 \end{aligned}$$
and so $c = (1, 2, 2)^T$. $Ux = c$ is the system:
$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + x_3 &= 2 \\ x_3 &= 2 \end{aligned}$$
Hence $x = (-1, 0, 2)^T$.