Calculus of Propositions $\mathcal{P}$ - Theorems

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1. Axiom Schema for $\mathcal{P}$.

   For any formulae φ, ψ, χ the following are axioms:

   A1. $\to \varphi \to \psi\varphi$

   A2. $\to\to \varphi \to \psi\chi \to\to \varphi\psi \to \varphi\chi$

   A3. $\to\to\to \varphi \perp\perp \varphi$

   The set of axioms for $\mathcal{P}$ is denoted $\mathcal{A}$.

2. In the metalanguage, the axioms may be written:

   (a) $\varphi \to (\psi \to \varphi)$

   (b) $(\varphi \to (\psi \to \chi)$
   $\to ((\varphi \to \psi) \to (\varphi \to \chi))$

   (c) $((\varphi \to\perp) \to\perp) \to \varphi)$
   (or equivalently $\neg\neg\varphi \to \varphi$.)

3. Remark. In the equivalent axiom system of Lukasiewicz (c) is replaced by

$$((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))$$

4. The set T of *theorems* of $\mathcal{P}$ is defined by:

   (a) $\mathcal{A} \subseteq T$

   (b) if $\varphi \in T$ and $\to \varphi\psi \in T$ then $\psi \in T$.

   (c) T is minimal with respect to (a) and (b).

5. If $\varphi \in T$ we write $\vdash \phi$.

6. A *proof* in $\mathcal{P}$ is a finite sequence (φ_i), $1 \leq i \leq n$ of formulae such that for each i either

   (a) φ_i is an axiom.

   (b) φ_i follows by modus ponens from two preceeding terms.

7. Definition. A set S of theorems is *inductive in* T if

   (a) $\mathcal{A} \subseteq S$.

   (b) $\varphi \in S$ and $\to \varphi\psi \in S \Rightarrow \psi \in S$.

8. Claim.

   Let $S \subseteq T$ is inductive in T then $S = T$.

   Proof.

   T is minimal containing the axioms and closed under modus ponens. Since $S \subseteq T$ it must be that $S = T$.

9. Claim. The result of a proof is a theorem. Specifically:

   If $\langle \varphi_0, \varphi_1, \ldots, \varphi_n \rangle$ is a proof then

$$\varphi_i \in T, \quad 1 \leq i \leq n$$

   Proof.

   By induction on n. If $n = 0$ then $\varphi_0 \in \mathcal{A}$ and so is a theorem. Let $n \in \mathbb{N}$ and consider a proof

$$\langle \varphi_0, \varphi_1, \ldots, \varphi_{n+1} \rangle.$$

   By induction hypothesis $\varphi_i \in T$ for $0 \leq i \leq n$. Since φ_{n+1} is either an axiom or obtained from previous formula by modus ponens, it must be that $\varphi_{n+1} \in T$.

10. Claim.

    The following are equivalent:

    (a) $\vdash \varphi$ (i.e., φ is a theorem).

    (b) φ has a proof.

    Proof.

    ($\Leftarrow$). Follows from the previous claim.

    $\Rightarrow$. Let

$$S = \{\varphi \in T : \varphi \text{ has a proof}\}$$

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Clearly $\mathcal{A} \subseteq S$. Suppose that $\varphi \in S$ and $\varphi \to \psi \in S$. Then there exist proofs:

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n = \varphi, \rangle$$
$$\langle \beta_1, \beta_2, \ldots, \beta_n = \varphi \to \psi, \rangle$$

Then

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n, \psi \rangle$$

is a proof of ψ i.e., $\psi \in S$. It follows that S is inductive in T and so $S = T$.

11. Definition.

    A subset $\Gamma \subseteq \mathcal{F}$. is called a *theory.*

12. The set $\overline{\Gamma} \subseteq \mathcal{F}$ of *theorems of* a theory Γ is defined by

    T1. $\mathcal{A} \cup \Gamma \subseteq \overline{\Gamma}$

    T2. $\phi \in \overline{\Gamma}$ and $\to \phi\psi \in \overline{\Gamma}$ imply $\psi \in \overline{\Gamma}$

    T3. $\overline{\Gamma}$ is minimal with respect to (a) and (b).

    Thus

    $$\overline{\Gamma} = \bigcap \{A \subseteq \mathcal{F} : T1 \text{ and } T2.\}$$

13. We write $\Gamma \vdash \phi$ instead of $\phi \in \overline{\Gamma}$

14. Exercise.

    Let $\phi \in \overline{\Gamma}$ and $\alpha \in \mathcal{V}$ and $H \in \mathcal{F}$. Show that $S^{\alpha}_{H}\phi \in \overline{\Gamma}$

15. Definition. A set $S \subseteq \overline{\Gamma}$ is *inductive in* Γ if

    (a) $\mathcal{A} \cup \Gamma \subseteq S$.

    (b) $\varphi \in S$ and $\to \varphi\psi \in S \Rightarrow \psi \in S$.

16. Claim.

    Let $S \subseteq$ is inductive in $\overline{\Gamma}$ then $S = \overline{\Gamma}$.

17. Example.

    Define

    (a) $\Gamma_0 = \Gamma \cup \mathcal{A}$

    (b) $\Gamma_{n+1} = \Gamma_n \cup \{\to \varphi\psi : \varphi, \psi \in \Gamma_n\}$

    Then
    $$\bigcup \{\Gamma_n :\in \mathbb{N}\}$$
    is inductive.

18. Deduction Theorem.

    Let $\Gamma \subseteq \mathcal{F}$, $\alpha, \beta \in \mathcal{F}$, then the following are equivalent:

    (a) $\Gamma \cup \alpha \vdash \beta$

    (b) $\Gamma \vdash \alpha \to \beta$

    (and so conditional proof can be dispensed with.)

    Proof. ($\Rightarrow$):

    Let

    $$S = \{\beta \in \overline{\Gamma \cup \alpha} : \Gamma \vdash \alpha \to \beta\}$$

    It is shown first that $S \supseteq \mathcal{A} \cup \Gamma \cup \alpha$. Let $\beta \in \mathcal{A} \cup \Gamma \cup \alpha$. If $\beta \in \mathcal{A} \cup \Gamma$. then

    $$\langle \beta, \beta \to \alpha \to \beta, \alpha \to \beta, \rangle$$

    is a proof of $\alpha \to \beta$ in Γ. If $\beta = \alpha$, then $\Gamma \vdash \alpha \to \beta$ follows from $\vdash \beta \to \beta$.

    It remains to show that S is closed under modus ponens. Suppose that $\mu, \mu \to \nu \in S$. Then

    (i) $\alpha \to \mu$ $\qquad$ $\mu \in S$

    (ii) $\alpha \to \mu \to \nu$ $\qquad$ $\mu \to \nu \in S$

    (iii) $(\alpha \to \mu \to \nu) \to ((\alpha \to \mu) \to (\alpha \to \nu)$ $\qquad$ A2.

    (iv) $(\alpha \to \mu) \to (\alpha \to \nu)$ $\qquad$ mp (ii),(iii).

    (v) $\alpha \to \nu$ $\qquad$ mp (i),(iv).

    and so $\Gamma \vdash \alpha \to \nu$. i.e., $\nu \in S$.

    Proof. ($\Leftarrow$):

    If $\Gamma \vdash \alpha \to \beta$ then $\langle \alpha, \alpha \to \beta, \beta \rangle$ is a proof of β in $\Gamma \cup \alpha$.