First Order Languages - Syntax

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1. In first-order logic the atomic formulas are predicates which express relationships between entities, thus making necessary the introduction of quantifiers.

2. Definition.

   A *signature* is a sequence

   $$\mathcal{L} = (\mathcal{R}, \mathcal{F}, \nu)$$

   where

   (a) $\mathcal{R}$ is a set $\{R_1, R_2 \ldots\}$ members of which are called *predicate symbols*.

   (b) $\mathcal{F}$ is a set $\{f_1, f_2 \ldots,\}$ members of which are called *function symbols*.

   (c) $\nu : \mathcal{R} \cup \mathcal{F} \to \mathbb{N}$ is the *arity* or number of arguments of each symbol. Function symbols of arity 1 are called *constants*.

   It is further required that $\mathcal{R} \cap \mathcal{F} = \emptyset$. Note that in some sources the binary relation symbol for equality $=$ is included in $\mathcal{R}$.

3. The *alphabet* of a first order language L contains the following symbols:

   (a) $v_0, v_1, \ldots.$ (*variables*);

   (b) $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$ (*not, and, or, if-then, if and only if*);

   (c) $\forall$, $\exists$ (*for all, there exists*);

   (d) $=$ (*equality symbol*);

   (e)), ((*parentheses*)

   (f) A *signature* $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \nu)$

   The symbols in (a)-(e), namely

   $$v_0, v_1, \ldots, \neg, \wedge, \vee, \to, \leftrightarrow, \forall, \exists, , =,), \ ($$

   are called *logical symbols* and those of the signature *non logical*. The set $\mathcal{R} \cup \mathcal{F}$ of non-logical symbols will be called the *symbol set* and will also be denoted $\mathcal{L}$.

4. Example.

   In set theory, the only nonlogical symbol is set membership $\in$.

5. Definition.

   Let $\mathcal{L}$ be a signature. The $\mathcal{L}$-*terms* are given recursively by

   (a) Every variable is a term.

   (b) Every constant is a term.

   (c) If the strings $t_1, t_2, \ldots t_n$ are terms and f is an n-ary function symbol then $f(t_0 \ldots t_n)$ is an $\mathcal{L}$-term.

   The set of $\mathcal{L}$-*terms* will be denoted $\mathcal{T}_{\mathcal{L}}$

6. Definition.

   The set of $\mathcal{L}$-formulas is defined by

   (a) If $t_1, t_2, \ldots t_n$ are $\mathcal{L}$-terms and $R \in \mathcal{R}$ is an n-ary relation symbol then

   $$Rt_1 \ldots t_n$$

   is an *atomic* formula. In particular $= t_1 t_2$ is atomic.

   (b) If ϕ, ψ are $\mathcal{L}$-formula then $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$ and $\phi \leftrightarrow \psi$ are $\mathcal{L}$-formulae.

   (c) If ϕ is an $\mathcal{L}$-formula and x is a variable then $\exists x\phi$ and $\forall x\phi$ are $\mathcal{L}$-formulae.

7. Hierachy of operations.

   $\neg$ is evaluated first. Then $\wedge$ and $\vee$ followed by quantifiers and finally $\to$.

8. Example.

   If v is a variable and f is a one place and g a two place function symbol, then v, $f(v)$, $g(v, f(v))$, $g(g(f(f(v)), v)$ etc. are terms.

[1] http://pennance.us –Partially based on a lecture by J. M. Lima

9. Definition.

   The set $V(t)$ of variables of a term $t \in \mathcal{T}$ is defined by:

   (a) If c is a constant, $V(c) = \emptyset$.

   (b) If x is a variable $V(x) = \{x\}$.

   (c) If f is an n-ary function,

   $$V(ft_1 \dots t_n) = V(t_1) \cup \dots \cup V(t_n)$$

10. Definition (Free Variables).

    The *free variables* of a formula F are then defined as follows:

    (a) Atomic formulae,

    $$V_f(Rt_1 \dots t_n) = V(t_1) \cup \dots \cup V(t_n).$$

    (b) Inductively for any formulae F, G and variable x:

    i. $V_f(\neg F) = V_f(F)$

    ii. $V_f(F \to G) = V_f(F) \cup V_f(F)$

    iii. $V_f(\forall x F) = V_f(F) \setminus \{x\}$

11. Definition.

    The *bound variables* of a formula F are then defined as follows:

    (a) For F is atomic, $V_b(F) = \emptyset$.

    (b) For any formulae F, G and variable x:

    i. $V_b(\neg F) = V_f(F)$

    ii. $V_b(F \to G) = V_b(F) \cup V_b(F)$

    iii. $V_b(\forall x F) = V_b(F) \cup \{x\}$

12. Variables If F is a formula the variables of F are given by

    $$V(F) = V_b(F) \cup V_f(F).$$

13. Example.

    Let φ be the formula

    $$\forall x \forall y (F(x) \to G(h(x), x, z)).$$

    Then $x, y \in V_b(\varphi)$ and $z \in V_f(\varphi)$. The variable w is neither bound nor free.

14. Example.

    In

    $$F(x) \to \forall x\, G(x).$$

    The first occurrence of x, i.e., that in $F(x)$ is free whereas the x in $\forall x\, G(x)$ is bound. Hence $x \in V_b(\varphi) \cap V_f(\varphi)$

15. Definition.

    A formula is called a *sentence* if $V_f = \emptyset$.

16. Remark.

    Sentences have well-defined truth values under an interpretation (see later).