Propositional Logic - Syntax

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1. Definition.

   The language $\mathcal{L}_\mathcal{P}$ of $\mathcal{P}$ has *symbol set*

   $$\mathcal{S} = \{\bot, p, *, \to\}$$

2. A finite sequence of letters in $\mathcal{S}$ is called a *word*. The set of words under concatenation is the free monoid on $\mathcal{S}$ and is denoted $\mathcal{S}^*$.

3. Logical Connectives.

   (a) $\bot$ is called *falsehood*.

   (b) $\to$ is called *implication*.

4. The set Var of *variables* is defined recursuvely as follows:

   (a) $p \in \text{Var}$

   (b) if $v \in \text{Var}$ then $v* \in \text{Var}$

   (c) Var is minimal with respect to (a) and (b).

5. The elements of $\text{At} := \text{Var} \cup \{\bot\}$ are called *atoms* or *atomic formulae.*

6. The set For of formulae is defined recursively by:

   (F_1) $\text{At} \subseteq \text{For}$

   (F_2) if $\phi, \psi \in \text{For}$ then $\to \phi\psi \in \text{For}$

   (F_3) For is minimal with respect to F_1 and F_2.

7. Corollary

   $$\text{For} = \bigcap\{A \subseteq \Sigma^* : F_1 \text{ and } F_2.\}$$

8. Definition.

   A set $S \subseteq \text{For}$ is called *inductive* if

   (a) $\{\bot, \text{Var}\} \subseteq S$

   (b) $\alpha, \beta \in S \Rightarrow \to \alpha\beta \in S$.

9. Claim. if S is inductive then $S = \text{For}$.

10. Equivalent definition of the formulae: Define $F_0 = \text{At}$ and for $n > 0$ let

    $$F_{n+1} = F_n \cup \{\to \alpha\beta : \alpha, \beta \in \text{For}\}$$

    then $\text{For} = \bigcup\{F_n : n \in \mathbb{N}\}$

11. Claim.

    Let X be a property of formulae. If

    (a) X holds for all $\phi \in F_0$.

    (b) If X holds for all $\phi \in F_{n+1}$ whenever it holds for all $\phi \in F_n$.

    Then, X holds for all $\phi \in \mathcal{F}$.

    Proof.

    Let

    $$S = \{n \in N : X \text{ holds } \forall \phi \in F_n\}$$

    Then $0 \in S$ and $n + 1 \in S$ whenever $n \in S$. By induction, $S = \mathbb{N}$ and so X holds for all $\phi \in \bigcup\{F_n : n \in \mathbb{N}\}$ Hence, X hold for all $\phi \in \text{For}$.

12. Metalinguistic Abbreviations:

$$
\begin{aligned}
p_0 &\equiv p \\
p_1 &\equiv p* \\
p_2 &\equiv p** \\
\vdots &\equiv \vdots \\
(\alpha \to \beta) &\equiv \to \alpha\beta \\
\alpha \to \beta &\equiv (\alpha \to \beta) \\
\neg\alpha &\equiv \alpha \to \bot \\
\top &\equiv \bot \to \bot \\
\alpha \to \beta \to \gamma \to \delta &\equiv (\alpha \to (\beta \to (\gamma \to \delta))) \\
\vee\alpha\beta &\equiv \neg\alpha \to \beta \\
\alpha \vee \beta &\equiv \vee\alpha\beta \\
\alpha \wedge \beta &\equiv \neg(\neg\alpha \vee \neg\beta) \\
\alpha \leftrightarrow \beta &\equiv (\alpha \to \beta) \to (\beta \to \alpha) \\
\alpha|\beta &\equiv \neg\alpha \vee \neg\beta
\end{aligned}
$$

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13. Hierarchy of operations.

    If no parentheses:

    (a) $\neg$ take precedence over $\rightarrow$.

    (b) Repetitions of connectives will be group to the right. For example we write $\alpha \rightarrow \beta \rightarrow \gamma$ instead of $(\alpha \rightarrow (\beta \rightarrow \gamma))$.