First Order Logic – Substitution

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1. Let V be the set of variable of a first order language and $\mathcal{T}$ the set of terms. Let $X \subseteq V$. A function $q \in \mathcal{T}^X$ is called a *term assignment*.

2. Notation.

   If $X = \{x_1, x_2, \ldots, x_n\}$ and $q : X \to \mathcal{T}$ is given by $q(x_i) = t_i, \quad 1 \le i \le n$ then we will write

   $$q = \begin{bmatrix} x_1 \cdots x_n \\ t_1 \ldots t_n \end{bmatrix}$$

3. Definition. (*Simultaneous Substitution*).

   Let $q : X \to \mathcal{T}$ a term assignment and let $t = q(x)$ and t' a term. The *substitution of t for x inside t'* (denoted $t' \begin{bmatrix} x \\ t \end{bmatrix}$ or by $t'(q)$) is defined by

   (a) If y is a variable

   $$y(q) = y \begin{bmatrix} x \\ t \end{bmatrix} = \begin{cases} t & \text{if } y \in X \ , \\ y & \text{if } y \neq X \end{cases}$$

   (b) If f is a function symbol or arity n, then

   $$ft_1 \ldots t_n(q) = ft_1(q) \ldots t_n(q)$$

4. Definition.

   The *free substitution of t for x inside a formula F (denoted by $F(q)$)* recursively:

   (a) $rt_1 \ldots t_n(q) = rt_1(q) \ldots t_n(q)$ where r is a relational symbol of order n.

   (b) $(\neg F)(q) = \neg F(q)$

   (c) $(F \to G)(q) = F(q) \to G(q)$

   (d) $(\forall x F)(q) = \forall x F(q')$ where

   $$q' = q - (x, q(x))$$

   i.e., q' is the restriction of q to $X' = X - x$

5. Alternative Notation.

   If $q(x_i) = t_i, \quad 1 \le i \le n$ then $F(q)$ is denoted:

   $$F(q) = F \begin{bmatrix} x_1 \cdots x_n \\ t_1 \cdots t_n \end{bmatrix}$$

6. Special Case.

   If $n = 1$, a single variable x is substituted by a term t,

   (a) If F atomic then $F \begin{bmatrix} x \\ t \end{bmatrix}$ is obtained from F by replacing x by t.

   (b) $(\neg F) \begin{bmatrix} x \\ t \end{bmatrix} = \neg F \begin{bmatrix} x \\ t \end{bmatrix}$

   (c) $(F \to G) \begin{bmatrix} x \\ t \end{bmatrix} = F \begin{bmatrix} x \\ t \end{bmatrix} \to G \begin{bmatrix} x \\ t \end{bmatrix}$

   (d) $(\forall x F) \begin{bmatrix} y \\ t \end{bmatrix} = \begin{cases} \forall x F & \text{if } y = x, \\ \forall x F \begin{bmatrix} y \\ t \end{bmatrix} & \text{if } y \neq x \end{cases}$

7. Example.

   Let term assignment $q : \{x_0, x_1\} \subseteq V \to \mathcal{T}$ be given by

   $$x_0 \mapsto x_1, \quad x_1 \mapsto x_0)$$

   i.e.

   $$q = \begin{bmatrix} x_0 \ x_1 \\ x_1 \ x_0 \end{bmatrix}$$

[1] http://pennance.us

Then if φ is the formula $x_0 \to (x_1 \to x_0)$

$$\begin{aligned} \varphi(q) &= [x_0 \to (x_1 \to x_0](q) \\ &= x_0(q) \to (x_1(q) \to x_0(q)) \\ &= x_1 \to (x_0 \to x_1) \end{aligned}$$

Remark. Simultaneous substition differs from successive substitution. For example

$$\begin{aligned} \varphi \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} &\equiv [x_0 \to (x_1 \to x_0)] \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \\ &= x_1 \to (x_1 \to x_1) \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \\ &= x_0 \to (x_0 \to x_0) \end{aligned}$$

8. Remark. Part 4(d) above ensures that only free variables are subject to change. If this restriction is ignored, as in

$$(\forall x\ x{+}y = y{+}x) \begin{bmatrix} x \\ z \end{bmatrix} = \forall x\ z{+}y = y{+}z$$

the semantic interpretation will be incorrect. For this reason the definition restricts substitution in $\forall x F$ to variables in $X - x$. Even this restriction is adhered to, semantic meaning can change under a substitution.

9. Example.

The substitution

$$(\forall x\ x{+}y = y{+}x) \begin{bmatrix} y \\ x \end{bmatrix} = \forall x\ x{+}x = x{+}x$$

is semantically incorrect. Even though y is free, it falls within the scope of the quantifier $\forall x$ and so if a term t containing x is substituted for y, a semantically incorrect formula will result. Hence, for logical deduction, a notion of *correct* or *proper* substition is required.

10. Definition. (Correct Substitution).

Let

$$q : X \subseteq V \to \mathcal{T}$$

be a term assignment and F a formula. The relation q is a *proper or correct substitution* in F (denoted $C(q,F)$) is defined by:

(a) If F is atomic, then $C(q,F)$.

(b) $C(q,F)$ $C(q,\neg F)$ if $C(q,F)$.

(c) $C(q,F \to G)$ if $C(q,F)$ and $C(q,G)$.

(d) $C(q,\forall x F)$ if

i. $C(q',F)$ and

ii. $x \notin V(T)$ where

$$T = q\left[X \cap V_f\,(\forall x\, F)\right]$$

Condition (ii) ensures that if v is free in $(\forall x\, F)$, then x is not a variable of the term $q(v)$ which is substituted for v.

11. Example. Let R be a binary relation symbol, f a binary function symbol and q the term assignment

$$q = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_0 & x_1 \end{bmatrix}$$

$$\begin{aligned} Rx_0fx_1x_2(q) &= Rx_0(q)fx_1(q)x_2(q) \\ &= Rx_2fx_3x_1 \end{aligned}$$

On the other hand, if

$$q = \begin{bmatrix} v_0 & v_2 \\ v_4 & fv_1v_2 \end{bmatrix}$$

then,

$$\begin{aligned} (\exists v_0 Rv_0fv_1v_2)(q) &= \exists v_0(Rv_0fv_1v_2)(q') \\ &= \exists v_0(Rv_0(q'))fv_1(q')v_2(q') \\ &= \exists v_0Rv_0)fv_1fv_1v_2 \end{aligned}$$

where $q' = q|_{\{v_2\}}$. Moreover (exercise) the substitution is correct. Notice that a possible interpretation of the statement before substitution is

$$\exists v_0 \; v_0 \geq v_1 + v_2$$

12. Example.

$$(\exists x \; x + x = y) \begin{bmatrix} y \\ x \end{bmatrix} = \exists x (x + x = x)$$

is improper since $x \in V(q(y)) = \{x\}$.

13. Example.

Let q the term assignment $\begin{bmatrix} x \\ y \end{bmatrix}$ with domain $D_q = \{x\}$. Consider the substitutions

(a) $(\forall x \, x + y = y + x)(q)$

(b) $(\forall y \, x + y = y + x)(q)$

In substitution (a) the domain of the restriction $q' = q - (x, q(x))$ is empty and no variables are eligible for substitution. The formula is unchanged.

In (b), $q' = q$ and the result of substitution is

$$\forall y \, y + y = y + y.$$

$$\begin{aligned} T &= q[D_q \cap V_f(\varphi)] \\ &= q[\{x\}] \\ &= \{y\} \end{aligned}$$

Since the quantified variable $y \in V(T)$ the substituion is not proper.

14. Example.

Let q the term assignment $\begin{bmatrix} x \\ y \end{bmatrix}$ and let φ be the formula $\forall y \, \forall x \; x + y = y + x)$ Noting that $q' = q - (y, q(y) = q$ yields:

$$\begin{aligned} \varphi(q) &\equiv (\forall y \, \forall x \; x + y = y + x)(q) \\ &\equiv \forall y \, (\forall x \; x + y = y + x)(q) \\ &\equiv \forall y \, \forall x \; y + y = y + y \end{aligned}$$

and it is easy to check that the substitutions is proper.

15. Example.

Let q the term assignment $\begin{bmatrix} x \\ y \end{bmatrix}$ and let ψ be the formula $\forall y \, \forall x \; x + y = y + x)$ Since the restriction $q - (x, q(x) = \emptyset$:

$$\begin{aligned} \psi(q) &\equiv (\forall x \, \forall y \; x + y = y + x)(q) \\ &\equiv \forall x \, (\forall y \; x + y = y + x)(\emptyset) \\ &\equiv \forall y \, \forall x \; y + y = y + y \end{aligned}$$

and the substitution is proper (exercise) and hence semantically correct.