Functions, Relations and Structures

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1. Definition.

   Let A be a set and $n \in \mathbb{N}$.

   (a) A subset $R \subseteq A^n$ is called an *n-ary relation* on A.

   (b) A function $f : A^n \to A$ is called an *n-ary operation* on A.

   In either case n is called the *arity*. A functions of arity 0 is called a *constant*.

2. Example.

   (a) $+ : \mathbb{N}^2 \to \mathbb{N}$ is a binary operation on $\mathbb{N}$.

   (b) Congruence modulo 3 is a binary relation on $\mathbb{N}$..

3. Definition.

   A *structure* is a triple (A, S_R, S_F) where S_R is a set of relations on A, S_F is a set of functions on A.

4. A structure is called

   (a) *pure* if $S_F = \emptyset$.
   (e.g. $(\mathbb{N}, \leq)$)

   (b) an *(universal) algebra* or *algebraic* if $S_R = \emptyset$
   (e.g. a group $(G, \star, -, e)$)

5. In universal algebra, the *signature* $\sigma = (S, \text{ar})$ of a structure consists of the set S of functions and relations of the structure, together with a function $\text{ar} : S \to \mathbb{N}$ mapping each member of S to its arity.

6. Let $(A, \mathcal{F})$ be an algebra. An algebra $(A', \mathcal{F}')$ is a *subalgebra* if

   (a) $A' \subseteq A$.

   (b) $\mathcal{F}' = \{f|_{A'^{\text{ar}(f)}} : f \in \mathcal{F}\}$

7. Let $(A, \mathcal{F})$ be an algebra and $H \subseteq A$. The subalgebra generated by H is the intersection of all subalgebras of A containing H.

8. Let $\mathcal{A}$, $\mathcal{B}$ be structures of the same signature σ. A map $h : A \to B$ is a σ *-homomorphism* if

   For every n-ary function symbol f of σ and elements $a_1, a_2, \ldots, a_n \in \mathcal{A}$:

   $h(f(a_1, \ldots, a_n)) = f(h(a_1), \ldots, h(a_n))$.

   and for every n-ary relation symbol R of σ and any elements $a_1, a_2, \ldots, a_n \in \mathcal{A}$:

   $(a_1, \ldots, a_n) \in R \Rightarrow (h(a_1), \ldots, h(a_n)) \in R$.

9. Let $\mathcal{A} = (A, f_1, f_2, \ldots, f_n)$ be an algebraic structure. the sequence

   $$\langle \text{ar}(f_1), \text{ar}(f_2, \ldots, \text{ar}(f_n) \rangle$$

   is called the *type* of $\mathcal{A}$.

10. Claim.

    Let $\tau = \langle n_1, n_2, \ldots n_k \rangle$ be a type. Then there exists an algebra $(A^*, s_1, s_2, \ldots, s_k)$ of type τ in which A^* is the set of words of an alphabet A.

    Proof.

    Define a set $\text{Op} = \{f_1, f_2, \ldots f_n\}$ of elements called *operations* and Var be any set such that $\text{Var} \cap \text{Op} = \emptyset$. The elements of Var are called *variables*. Let $A = \text{Var} \cup \text{Op}$ and A^* the set of words in A. For each i, define

    $$s_i : (A^*)^{n_i} \to A^*$$

    by

    $$s_i(a_1, a_2, \ldots a_k) = f_i a_1, a_2, \ldots a_k$$

    Then $\langle A^*, (s_i : 1 \leq i \leq k), \rangle$ is a structure of type τ.

11. The subalgebra T generated by Var is called the algebra of *terms*. It follows that T is the smallest set of words containing the variables and closed under the operations .

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12. Example.

To construct an algebra of terms of type $\tau = \langle 2, 0, 1, 2, \rangle$ let $\text{Op} = \{+, 0, -, *\}$ and $\text{Var} = \{x, y\}$. Let $A = \text{Op} \cup \text{Var}$. Define functions

$$\begin{aligned} s_1(aa') &= +aa', \quad a, a' \in A^* \\ s_2(\emptyset) &= 0 \\ s_3(a) &= -a, \quad a \in A^* \\ s_4(aa') &= *aa', \quad a, a' \in A^* \end{aligned}$$

Then $(A^*, (s_i : 1 \le i \le 4))$ is a structure of type τ. Since the sub-algebra of terms T generated by Var is closed under the s_i

$$\begin{aligned} s_2(\emptyset) &= 0 \in T \\ s_3(y) &= -y \in T \\ s_3(x) &= -x \in T \\ s_4(xy) &= *xy \in T \\ s_3(*xy) &= - * xy \in T \\ &etc. \end{aligned}$$

13. Claim.

If t is a term, then either $t \in \text{Var}$ or $t = ft_1t_2 \dots t_n$ where f is an n-ary operation and $t_1, t_2, \dots t_n \in T$.

Proof. Let

$$S = \{t \in T : t \in \text{Var} \text{ or } t = ft_1t_2 \dots t_n\}$$

where $t_1, t_2, \dots t_n \in T$ and $ar(f) = n$. Then S contains Var and is closed under operations. But T is minimal with these properties so that $T \subseteq S$. It follows that $S = T$.

14. Definition.

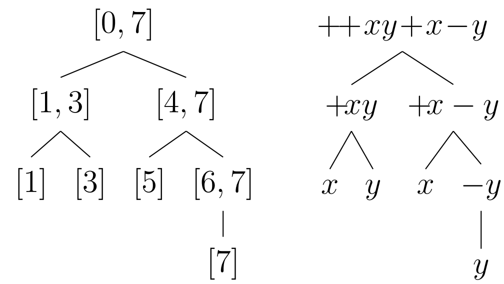
Let $a = a_0a_1 \dots a_n$ be a term. A segment $[i, j] \subseteq \{0, 1, 2, \dots n]\}$ is called an *occurrence* of the word $a_ia_{i+1} \dots a_j$ in a. The set of all occurrences of terms in a is denoted by $T[a]$.

15. Example.

Let

$$a = {+}{+}xy{+}x{-}y.$$

Then, the digraph $(T[a], \subseteq)$ is a directed tree:



16. Lemma.

A proper initial segment of a term is not a term. i.e., If $ab \in T$ and $a \in T$ then $b = \emptyset$.

Proof.

Suppose to the contrary that there exists a term a and a non empty word b such that $ab \in T$. Then, there exist an n-ary operation f and terms t_i and t_i' such that

$$\begin{aligned} a &= ft_1 \cdots t_n \\ ab &= ft_1' \dots t_n' \end{aligned}$$

It follows that

$$t_1 \dots t_n b = t_1' \dots t_n'$$

If a is a counter example of minimum length, it must be that $|t_1'| = |t_1|$ for if $|t_1'| > |t_1|$ then $t_1' = t_1 d$ where $d \neq \emptyset$ contradicting the minimality of a. By induction, $t_j = t_j'$, $1 \le j \le n$ and hence $b = \emptyset$ which is a contradiction.

17. Corollary [Unique readability for terms.]

If

$$ft_1t_2 \dots t_n = f't_1't_2' \dots t_m'$$

are terms then $n = m$ and $t_i = t_i'$, $1 \le i \le n$.