# Predicate Logic - Semantics

Prof. Philip Pennance[1] -Version: 2004.

1. A formal language $\mathcal{L}$ contains both logical and non-logical symbols. The non-logical symbols of $\mathcal{L}$ are specified by a signature. The definition of signature in this context differs from that previously given in algebraic structures. In what follows, the language and its signature will be denoted by the same letter.

2. A *signature* for a formal language $\mathcal{L}$ is a triple $\mathcal{L} = (S_{\text{Fun}}, S_{\text{Rel}}, \text{ar})$, where
   (a) $S_{\text{Fun}}$ is a set of elements called *function symbols*.
   (b) $S_{\text{Rel}}$ is a set of elements called *relation symbols*
   (c) $S_{\text{Fun}} \cap S_{\text{Rel}} = \emptyset$
   (d) Neither of $S_{\text{Fun}}$, $S_{\text{Rel}}$ contain logical symbols.
   (e) ar is a function:
   $$ar : S_{\text{Fun}} \cup S_{\text{Rel}} \to \mathbb{N}$$
   which assigns an arity to every function and relation symbol.

   The set $S = S_{\text{Fun}} \cap S_{\text{Rel}}$ is called the *symbol set*.

3. The usual signature $\mathcal{L}_{ag}$ of abelian groups has
   (a) $S_{\text{Fun}} = \{+, -, 0, \}$
   (b) $S_{\text{Rel}} = \emptyset$

   where $\text{ar}(0) = 0$, $\text{ar}(-) = 1$, and $\text{ar}(+) = 2$.

4. The signature of sets is empty set.

5. Definition.
   Let $\mathcal{L}$ be a signature. A $\mathcal{L}$*-structure* is a pair $\mathcal{E} = (E, \cdot)$ where
   (a) $E$ is a non-empty set called the *domain*.
   (b) $\cdot$ is a map defined on the symbol set called the *interpretation function*, which which ascribes:
      i. to every $n$-ary relation symbol $R \in \mathcal{R}$ , an $n$-ary relation $\dot{R}$ on $E$.
      ii. to every $n$-ary function symbol $f \in \mathcal{F}$, an $n$-ary function $\dot{f}$ on $E$.

6. Notational convention.
   If a signature $\mathcal{L}$ has symbol set
   $$S_{\mathcal{L}} = \{R_1, R_2, \ldots f_1, f_2, \ldots\}$$
   an $\mathcal{L}$ structure with domain domain $E$ is often denoted:
   $$\mathcal{E} = \langle E, \dot{R}_1, \dot{R}_2, \ldots, \dot{f}_1, \dot{f}_2, \ldots\rangle$$
   with arity suppressed.

7. Example.
   Peano arithmetic has symbol set $\mathcal{L}_{\mathbb{N}}$ has symbol set
   $$\mathcal{S}_{\mathbb{N}} = \langle +, s, 0\rangle$$
   A possible structure is:
   $$\mathcal{E}_{\mathbb{N}} = \langle \mathbb{N}, \dot{+}, \dot{s}, \dot{0}\rangle$$
   where
   $$\dot{+} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
   $$\dot{s} : \mathbb{N} \to \mathbb{N}$$
   $$\dot{0} \in \mathbb{N}$$

[1] http://pennance.us

8. Example. $\mathcal{E}_{\mathcal{G}} = (V, E)$ where $E$ is a binary relation of $V$ is a structure for graph theory.

9. Definition.

   An $\mathcal{L}$-*interpretation* $\mathcal{I}$ is a pair $(\mathcal{E}, \sigma)$ where $\mathcal{E}$ is an $\mathcal{L}$-structure and $\sigma \in E^{\text{Var}}$. The map $\sigma$ called an $\mathcal{E}$-assignment. The extension of $\sigma$ to a map on terms

   $$[t \mapsto t^{\mathcal{I}}] \in E^{Term}$$

   defined by:

   (a) If $t = v$ is a variable $v^{\mathcal{I}} = \sigma(v)$.

   (b) If $t = ft_1t_2 \ldots t_n$ where $f$ is an $n$-ary function symbol then

   $$t^{\mathcal{I}} = \dot{f}t_1^{\mathcal{I}}t_2^{\mathcal{I}} \ldots, t_n^{\mathcal{I}}$$

   is called a *valuation.*

10. Interpreting Equality.

    To avoid treating equality as a binary operation which entails its inclusion in the signature and the addition of axioms about equality, so called *normal models* treat $=$ as a logical constant and $\doteq$ as the real equality relation in any interpretation. (See Wikipedia for details).

11. Definition.

    An $\mathcal{E}$-assignment $\sigma$, an element $a \in E$ and a variable $x$ together determine another $\mathcal{E}$-assignment $\sigma_a^x \in E^{\text{Var}}$ given by

    $$\sigma_a^x(y) = \begin{cases} a & \text{if } y = x \ , \\ \sigma(y) & \text{if } y \neq x \end{cases}$$

    i.e., $\sigma_a^x = \sigma + (x, a) - (x, \sigma(x))$

12. Tarski's definition of truth

    Let $\mathcal{I} = (\mathcal{E}, \sigma)$ be an interpretation and $\phi$ a formula. The relation $\phi$ *is true* in $\mathcal{I}$ or $\mathcal{I}$ is a *model* of $\phi$, also called the *satisfaction* relation, writen $\mathcal{I} \models_\sigma \phi$,is defined by induction on $\phi$ by:

    (a) If $\phi = Rv_1v_2 \ldots v_k$ is atomic,

    $$\mathcal{I} \models_\sigma \phi \iff \dot{R}v_1^\sigma v_2^\sigma \ldots v_k^\sigma$$

    (b) If $\phi$ is constructed using logical connectives (i.e., in the metalanguage) from simpler formulas then the truth of $\mathcal{I} \models_\sigma \phi$ is determined by the truth tables. (See examples below).

    (c) If $\phi = \forall x A$ or $\phi = \exists x A$, then

    $$\mathcal{I} \models_\sigma \forall x A \iff \forall a \in E, \quad \mathcal{I}_a^v \models_{\sigma_a^v} A$$
    $$\mathcal{I} \models_\sigma \exists x A \iff \exists a \in E, \quad \mathcal{I}_a^v \models_{\sigma_a^v} A$$

    where $\mathcal{I}_a^v = (\mathcal{E}, \sigma_a^v)$.

13. Alternative notations:

    $\mathcal{I} = (\mathcal{E}, \sigma)$ be an interpretation and $\phi$ a formula. The following are synonymous:

    (a) $\mathcal{I} \models_\sigma \phi$

    (b) $(\mathcal{E}, \sigma) \models \phi$

    (c) $\mathcal{E} \models_\sigma \phi$

14. Example of 10(c).

    Let $R$ be a relational symbol of arity 3. The statement

    $$\mathcal{I} \models_\sigma \forall x_1 R x_1 x_2 x_3$$

    should mean

    $$\forall a \in E \ \dot{R}\, a \sigma(x_2) \sigma(x_3) \qquad (1)$$

    This is not in the recursive form of 10(a). However,

    $$\sigma_a^{x_1}(x_1) = a$$
    $$\sigma_a^{x_2}(x_2) = x_2$$
    $$\sigma_a^{x_3}(x_3) = x_3$$

    and so

    $$(1) \equiv \forall a \in E \ \dot{R}\sigma_a^{x_1}(x_1)\sigma_a^{x_2}(x_2)\sigma_a^{x_3}(x_3)$$
    $$\equiv \forall a \in E \ \mathcal{I}_a^{x_1} \models R x_1 x_2 x_3$$

    which has the required recursive form.

15. Examples of 10(b).

    (a) $\mathcal{I} \models_\sigma (\neg A)$ if and only if $\neg \mathcal{I} \models_\sigma A$

    (b) $\mathcal{I} \models_\sigma (A \vee B)$ iff either $\mathcal{I} \models_\sigma A$ or $\mathcal{I} \models_\sigma B$

16. Example. [Arithmetic]

    Let $\mathcal{I} = (\mathcal{E}_\mathbb{N}, \sigma)$ be the interpretation of arithmetic:

$$\mathcal{E}_\mathbb{N} = \langle \mathbb{N}, \dot{+}, \dot{\times}, \dot{0}, \dot{1} \rangle$$

where $\sigma \in \mathbb{N}^{\text{Var}}$ is the map $x_n \mapsto 2n$. Then the following are interpretations of formulae:

| $\varphi \in$ For | Interpretation |
|---|---|
| $x_1 \times x_2 = x_3$ | $2 \times 4 = 6$ |
| $\exists x_0, x_0 + x_0 = x_1$ | $\exists n \in \mathbb{N}, n + n = 2$ |
| $\exists x_1, x_0 = x_1$ | $\exists n \in \mathbb{N}, 0 = n$ |
| $\forall x_0 \; \exists x_1, x_0 \leq x_1$ | $\forall n \in \mathbb{N} \; \exists m \in \mathbb{N}, n \leq m$ |

17. Example.

    Let $\mathcal{L}$ be a signature, $\mathcal{E} = (E, \cdot)$ an $\mathcal{L}$-structure and $\mathcal{I} = (\mathcal{E}, \sigma)$ an interpretation in which

$$x_1 \mapsto a_1, \quad x_2 \mapsto a_1 \quad x_3 \mapsto a_3$$

Then if $\varphi$ is the formula

$$\forall x_1 \;\; x_1 = x_3$$

$$\begin{aligned} \mathcal{I} \models_\sigma \varphi &\equiv \forall a \in E \; \mathcal{I}_a^{x_1} \varphi \\ &\equiv \; \forall a \in E \;\; \mathcal{I}_a^{x_1} = x_1 x_3 \\ &\equiv \; \forall a \in E \quad = \sigma_a^{x_1}(x_1) \sigma_a^{x_1}(x_3) \\ &\equiv \; a_1 = a_3 \; \wedge a_2 = a_3 \; \wedge a_3 = a_3 \end{aligned}$$

18. Example.

    Let $\mathcal{I} = (\mathcal{E}_{ag}, \sigma)$ be the interpretation of abelian group theory with

$$\mathcal{E}_{ag} = \langle \mathbb{R}, \dot{+}, \dot{0}, \rangle$$

and $\sigma \in \mathbb{R}^{\text{Var}}$ the assignment $\sigma(x) = \pi$ for all $x \in$ Var. If $\varphi \in \text{For}_{gp}$ is the axiom

$$\forall x \; x + 0 = x.$$

Then,

$$\begin{aligned} \mathcal{I} \models_\sigma \varphi &\equiv \forall r \in \mathbb{R} \;\; \mathcal{I}_r^x \varphi \\ &\equiv \forall r \in \mathbb{R} \;\; \mathcal{I}_r^x \; x + \dot{0} = x \\ &\equiv \forall r \in \mathbb{R} \;\; \mathcal{I}_r^x \; x + \dot{0} = x \\ &\equiv \forall r \in \mathbb{R} \;\; \sigma_r^x(x) \dot{+} \dot{0} = \sigma_r^x(x) \\ &\equiv \forall r \in \mathbb{R} \;\; r + \dot{0} = r \end{aligned}$$

19. Coincidence Lemma.

    $\mathcal{I} = (\mathcal{E}, \sigma)$ and $\mathcal{I}' = (\mathcal{E}, \sigma')$ be two interpretations and $\phi$ a formula. If

$$\sigma|_{\text{Var}\,\phi} = \sigma'|_{\text{Var}\,\phi}$$

then following are equivalent:

   (a) $\mathcal{I} \models \phi$

   (b) $\mathcal{I}' \models \phi$

Proof. Exercise.

20. Corollary.

    If $V_{free}(\phi) = \emptyset$ then either

    (a) $\mathcal{E} \models_\sigma \phi$ for all formulae $\phi$

    (b) $\mathcal{E} \not\models_\sigma \phi$ for all formulae $\phi$

21. Example.

    Let $\varphi$ be the formula $\forall x \; x = c$. Then

$$\begin{aligned} \mathcal{I} \models_\sigma \varphi &\equiv \forall a \in E \;\; \sigma_a^x x = \sigma_a^x c \\ &\equiv \forall a \in E \;\; a = c \end{aligned}$$

In this case $V_{free}\varphi = \emptyset$ and $\mathcal{I} \models_\sigma \varphi$ is independent of $\sigma$.

22. Claim.

    If $x \notin V_{free}(\phi)$ then for any valuation $\sigma \in E^{\text{Var}}$ the following are equivalent:

    (a) $\mathcal{I} \models_\sigma \forall x \; \varphi$

    (b) $\mathcal{I} \models_\sigma \varphi$

Proof.

Since $x \notin \mathrm{Var}_{free}\,\varphi$, Both $\sigma$ and $\sigma_a^x$ coincide on $\mathrm{Var}_{free}\,\varphi$. Hence, using the coincidence lemma:

$$\begin{aligned}\mathcal{I} \models_\sigma \forall x\varphi &\equiv \forall a \in E \;\; \mathcal{I} \models_{\sigma_a^x} \varphi \\ &\equiv \forall a \in E \;\; \mathcal{I} \models_\sigma \varphi \\ &\equiv \forall a \in E \;\; \mathcal{I} \models_\sigma \varphi\end{aligned}$$

But $\mathcal{I} \models_\sigma \varphi$ is independent of $\sigma$ and so

$$\forall a \in E \;\; \mathcal{I} \models_\sigma \varphi \iff \mathcal{I} \models_\sigma \varphi$$

23. Corollary.

Let $s$ and $t$ be terms and $x$ is not a variable of either. then:

$$\begin{aligned}\mathcal{I} \models_\sigma \forall x \; s = t \;&\equiv\; \mathcal{I} \models_\sigma s = t \\ &\equiv\; \sigma(s) = \sigma(t)\end{aligned}$$