Propositional Calculus – Semantics

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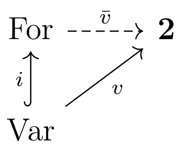
1. Let $2 = \{0, 1\}$ and let Var be the set of variables. A function $v \in 2^{\text{Var}}$ extends to a *truth assignment* $\bar{v}$: For $\to 2$



defined on atomic formula by

$$\bar{v}(\beta) = \begin{cases} v(\beta) & \text{if } v \in \text{Var} \;, \\ 0 & \text{if } v = \perp \end{cases}$$

and recursively on formulae by

$$\bar{v}(\to \varphi\psi) = \max\{1 - v(\varphi), v(\psi)\}$$

2. Corollary.

   (a) $\bar{v}(\neg\varphi) = 1 - v(\varphi)$.

   (b) $\bar{v}(\varphi \vee \psi) = \max\{v(\varphi), v(\psi)\}$.

   (c) $\bar{v}(\varphi \wedge \psi) = \min\{v(\varphi), v(\psi)\}$.

   (d) $\bar{v}(\varphi \leftrightarrow \psi) = 1$ iff $v(\varphi) = v(\psi)$

3. Definition.

   A formula φ is

   (a) *tautology* if

   $$\bar{v}(\varphi) = 1 \quad \forall v \in 2^{\text{Var}}.$$

   (b) a *contradiction* if

   $$\bar{v}(\varphi) = 0 \quad \forall v \in 2^{\text{Var}}.$$

4. Notation.

   If ϕ is a tautology We write

   $$\models \varphi.$$

5. φ is said to be *logically equivalent* to ψ, if $\models \varphi \leftrightarrow \psi$.

6. Definition.

   Let φ be a formula with variables

   $$\text{Var}_\varphi = \{p_1, p_2, \ldots, p_k\}.$$

   and Let

   $$v_1, v_2, \ldots v_{2^k} \in 2^{\text{Var}}$$

   be any truth assignments with

   $$\begin{aligned} v_1|_{p_1p_2\ldots p_n} &= (\overbrace{1, 1, 1, \cdots, 1}^{\text{k}}) \\ v_2|_{p_1p_2\ldots p_n} &= (1, 1, 1, \cdots, 0) \\ \vdots \quad &= \quad \vdots \\ v_{2^k}|_{p_1p_2\ldots p_n} &= (0, 0, 0, \cdots, 0) \end{aligned}$$

   The *truth table* for φ is the sequence in B^{k+1} with terms:

   $$\begin{aligned} \varphi_1 &= (\overbrace{1, 1, 1, \cdots, 1}^{\text{k}}, \bar{v}_1(\varphi)) \\ \varphi_2 &= (1, 1, 1, \cdots, 0, \bar{v}_2(\varphi)) \\ \vdots \quad &= \quad \vdots \\ \varphi_{2^k} &= (0, 0, 0, \cdots, 0, \bar{v}_{2^k}(\varphi)) \end{aligned}$$

   The *truth function* determined by φ is the map

   $$T^\varphi : B^k \to B$$

   given by

   $$T^\varphi(v_i(p_1), v_i(p_2), \ldots, v_i(p_k) = \bar{v}_i(\varphi)$$

   for $1 \leq i \leq 2^k$. Notice that the the graph of the truth function is just the set of terms of the truth table.

7. Example.

   Let $\varphi = p_3 \to p_4$. Let v_1, v_2, v_3, v_4 be truth assignments such that

   $$\begin{aligned} v_1|_{\text{Var}_\varphi} &= (1, 1) \\ v_2|_{\text{Var}_\varphi} &= (1, 0) \\ v_3|_{\text{Var}_\varphi} &= (0, 1) \\ v_4|_{\text{Var}_\varphi} &= (0, 0) \end{aligned}$$

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then

$$\begin{aligned}\bar{v}_2(\varphi) &= v_2(p_3 \to p_4)\\ &= \max\{1 - v_2(p_3), v_2(p_4)\}\\ &= \max\{0, 0\}\\ &= 0\end{aligned}$$

Similarly

$$\bar{v}_1(\varphi) = \bar{v}_3(\varphi) = \bar{v}_4(\varphi) = 1$$

8. Claim.

   If φ and ψ are logically equivalent then $T^\varphi = T^\psi$.

   Proof,

   $\models \varphi \leftrightarrow \psi$ means that $v(\varphi) = v(\psi)$ for all truth functions v.

9. Claim [1.9 Hamilton]

   If $\models \varphi$ and $\models \varphi \to \psi$ then $\models \psi$.

   Proof. Suppose not. Then there exists a truth function v such that $v(\psi) = 0$ and $v(\varphi) = 1$ and $v(\varphi \to \psi) = 1$. But

   $$\begin{aligned}v(\varphi \to \psi) &= \max\{1 - v(\varphi), v(\psi)\}\\ &= 0\end{aligned}$$

   which is a contradiction.

10. Definition (Substitution.)

    Let $H \in$ For and $\alpha \in$ Var. The substitution of α by H is defined on atoms by

    $$S^\alpha_H \beta = \begin{cases}\beta & \text{if } \beta \neq \alpha\ ,\\ H & \text{if } \beta = \alpha\end{cases}$$

    for any $\beta \in$ At, and recursively on formulae by

    $$S^\alpha_H (\to FG) = \to S^\alpha_H F\ S^\alpha_H G$$

    It follows by induction that for any formula F, $S^\alpha_H F \in$ For.

11. Definition.

    Let $v \in 2^{\mathrm{Var}}$ be a truth assignment. If $\alpha \in$ Var and $a \in \{0.1\}$ the truth assignment $v - (\alpha, v(\alpha)) + (\alpha, a)$, obtained by changing the value of v at α to a, is denoted $v \begin{bmatrix}\alpha\\ a\end{bmatrix}$. i.e.:

    $$v \begin{bmatrix}\alpha\\ a\end{bmatrix} (\beta) = \begin{cases}\beta & \text{if } \beta \neq \alpha\ ,\\ a & \text{if } \beta = \alpha\end{cases}$$

12. Example.

    Let F be the formula $p_0 \to p_1$ and v a truth assignment, say

    $$v = \begin{bmatrix}p_0 & p_1 & p_2 & \cdots\\ 1 & 0 & 1 & \cdots\end{bmatrix}$$

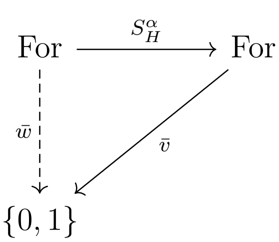
    Notice

    $$\begin{aligned}\bar{v} \circ S^{p_2}_{p_1} F &= \bar{v}(p_0 \to p_2)\\ &= 1\\ &= v \begin{bmatrix}p_1\\ v(p_2)\end{bmatrix} F\end{aligned}$$

    This is a special case of:

13. Claim.

    Let $v \in 2^{\mathrm{Var}}$, $\alpha \in$ Var and $H \in$ For . If $w = v \begin{bmatrix}\alpha\\ \bar{v}(H)\end{bmatrix}$. Then there $\bar{w} = \bar{v} \circ S^\alpha_H$

    Proof.

    Proceeding by induction, let

    $$S = \{F \in \text{For} : \bar{w}(F) = \bar{v} \circ S^\alpha_H(F)\}$$

    Notice, $\bar{w}(\bot) = \bar{v} \circ S^\alpha_H(\bot) = 0$ and so $\bot \in S$. Next, if F is a variable then

    $$\begin{aligned}\bar{w}(F) &= v \begin{bmatrix}\alpha\\ \bar{v}(H)\end{bmatrix} (F)\\ &= \begin{cases}\bar{v}(F) & \text{if } F \neq \alpha\ ,\\ \bar{v}(H) & \text{if } F = \alpha\end{cases}\\ &= \bar{v}\left(S^\alpha_H(F)\right)\end{aligned}$$

Hence At $\subseteq S$. It remains to prove closure of S under implication. Let $\theta, \psi \in S$. Then

$$\begin{aligned}\bar{w}(\rightarrow \theta\psi) &= \max\{1 - \bar{w}(\theta), \bar{w}(\phi)\} \\ &= \max\{1 - \bar{v} \circ S^{\alpha}_{H}(\theta), \bar{v} \circ S^{\alpha}_{H}(\phi)\} \\ &= \bar{v}\left(S^{\alpha}_{H}\theta \rightarrow S^{\alpha}_{H}\phi\right) \\ &= \bar{v} \circ S^{\alpha}_{H}\left(\theta \rightarrow \phi\right)\end{aligned}$$

14. Corollary.

If $F, H \in$ For and $\alpha \in$ Var,

$$\models F \implies \models S^{\alpha}_{H} F$$

Proof.

Suppose $\models F$. Then

$$\begin{aligned}v\left(S^{\alpha}_{H} F\right) &= \bar{w}(F) \\ &= 1\end{aligned}$$

15. Remark.

It follows from the discussion above that if $F \in$ For has variable $p_1, p_2, \ldots, p_n$ and $v \in 2^{\text{Var}}$ is a truth assignment then the truth of a substitution is determined from the truth function for F using:

$$v(S^{p_k}_{H} F) = T^{F}(v(p_1), \ldots, v(H), \ldots v(p_n))$$