First Order Languages - Axioms

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Axioms.

If φ is a formula belonging to one of the following three groups then φ is an axiom (written $\varphi \in \mathcal{A}$) and so to is any generalization $\forall x_1 \forall x_2 \ldots \forall x_n \varphi$ of φ.

I. Tautologies

(a) $\phi \to (\psi \to \phi)$

(b) $[\phi \to (\psi \to \chi)] \to [(\phi \to \psi) \to (\psi \to \chi)]$

(c) $(\neg\psi \to \neg\phi) \to (\phi \to \psi)$

II. Quantification.

(a) If the substitution is correct:

$$\forall x\ \phi \to \phi \left[\frac{x}{t}\right]$$

(b) Distributivity of $\forall$

For any variable x and any formulae ϕ and ψ,

$$\forall x(\phi \to \psi) \to (\forall x\phi \to \forall x\psi)$$

(c) Addition of a redundant quantifier

If x is not free in ϕ :

$$\phi \to \forall x\phi$$

III. Equality.

(a) For any term t

$$t = t$$

(b) For any ϕ, ϕ' atomic where ϕ' is obtained from ϕ by replacing t in zero or more places by s.

$$t = s \to (\phi \to \phi')$$

Theorems.

1. Let Σ any set of formulae. The set of *theorems* of Σ denoted $\overline{\Sigma}$ is defined as in propositional calculus by:

   (a) $\mathcal{A} \cup \Sigma \subseteq \overline{\Sigma}$

   (b) If ϕ and $\phi \to \psi$ are theorems then so is ψ.

   (c) $\overline{\Sigma}$ is minimal with respect to (1) and (2).

2. The Deduction relation.

$$\Sigma \vdash \phi \iff \phi \in \overline{\Sigma}$$

3. Deduction Theorem

$$\Gamma, \varphi \vdash \psi \iff \Gamma \vdash \varphi \to \psi.$$

   Proof. The same as in the calculus of propositions.

[1] http://pennance.us