Calculus of Propositions $\mathcal{P}$ - Simple Proofs

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1. Axiom Schema for $\mathcal{P}$.

   For any formulae F, G, H the following are axioms:

   A1. $\rightarrow F \rightarrow GF.$

   A2. $\rightarrow\rightarrow F \rightarrow GH \rightarrow\rightarrow FG \rightarrow FH.$

   A3. $\rightarrow\rightarrow\rightarrow F \perp\perp F.$

   The set of axioms for $\mathcal{P}$ is denoted $\mathcal{A}$.

2. Axiom Schema in the metalanguage.

   A1. $F \rightarrow (G \rightarrow F).$

   A2. $(F \rightarrow (G \rightarrow H) \rightarrow ((F \rightarrow G) \rightarrow (F \rightarrow H)).$

   A3. $((F \rightarrow \perp) \rightarrow \perp) \rightarrow F$ (i.e., $\neg\neg F \rightarrow F.$)

   Remark. In the Lukasiewicz schema (A3) is replaced by $((\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)).$

3. Claim I. $\vdash F \rightarrow F.$

   Proof – In the metalanguage:

(i)	$(F \rightarrow (G \rightarrow F)) \rightarrow ((F \rightarrow G) \rightarrow (F \rightarrow F))$	A2.
(ii)	$(F \rightarrow (G \rightarrow F)$	A1.
(iii)	$((F \rightarrow G) \rightarrow (F \rightarrow F))$	(i),(iii), mp
(iv)	$((F \rightarrow (G \rightarrow F)) \rightarrow (F \rightarrow F))$	inst. (iii)
(v)	$F \rightarrow F$	ii, iv, mp

   Proof – In the language $\mathcal{L}_P$

   (i) $\rightarrow\rightarrow F \rightarrow GF \rightarrow\rightarrow FG \rightarrow FF$

   (ii) $\rightarrow F \rightarrow GF$

   (iii) $\rightarrow\rightarrow FG \rightarrow FF$

   (iv) $\rightarrow\rightarrow F \rightarrow GF \rightarrow FF$

   (v) $\rightarrow FF$

   Remark. This theorem follows immediately from the Deduction Theorem.

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4. Claim II. $\vdash \bot \to F$

Proof. (Redundant lines are included for clarity.)

(i)	$((F \to \bot) \to \bot) \to F$	A3,
(ii)	$\neg\neg F \to F$	(i) repeated.
(iii)	$(\neg\neg F \to F) \to \bot \to (\neg\neg F \to F)$	A1.
(iv)	$\bot \to \neg\neg F \to F$	(iii)
(v)	$(\bot \to \neg\neg F \to F) \to ((\bot \to \neg\neg F) \to (\bot \to F))$	A2.
(vi)	$(\bot \to \neg\neg F) \to (\bot \to F)$	(iv), (v), mp.
(vii)	$\bot \to (F \to \bot) \to \bot$	A1.
(viii)	$\bot \to \neg\neg F$	(vii) repeated.
(ix)	$\bot \to F$	(vi), (viii), mp.

5. Claim III. $\vdash (F \to \bot) \to (F \to G)$ (Duns Scotus)

Proof.

(i)	$(\bot \to G) \to F \to (\bot \to G)$	A3.
(ii)	$(\bot \to G)$	Claim II.
(iii)	$F \to (\bot \to G)$	(i),(ii), mp.
(iv)	$(F \to (\bot \to G)) \to ((F \to \bot) \to (F \to G))$	A2.
(v)	$(F \to \bot) \to (F \to G)$	(iii),(iv), mp.