The Isoperimetric Inequality

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1. Introduction.

Isoperimetry is the study of figures that have perimeters of equal length. The classical isoperimetric inequality, proven here, is an algebraic expression of the fact that among all sufficiently nice curves of the same length the circle encloses the maximum area.

2. Lemma.

Let

$$\alpha(s) = (x(s), y(s)), \quad a \leq s \leq b$$

be a simple, plane, closed, regular curve. Then there exists a circle with parametrization (not necessarily simple)

$$\alpha^*(s) = (x(s), y^*(s)), \quad a \leq s \leq b$$

with $y^* \in C^1$.

Proof.

Without loss of generality assume that α has positive orientation. Since

$$x : [a, b] \to \mathbb{R}$$

is bounded it achieves a maximum and a minimum. Let m be the first minimum point of x. By a time translation it can be assumed that the first maximum of x occurs at $s = 0$. By a translation and scaling of α we can assume $x(0) = 1$ and $x(m) = -1$. Let

$$y^*(s) = \begin{cases} \sqrt{1 - x(s)^2)} & \text{if } 0 \leq s \leq m \text{ ,} \\ -\sqrt{1 - x(s)^2)} & \text{if } m < s \leq 1. \end{cases}$$

If $s \neq m$, then y^* is clearly differentiable.

If $s = m$,

$$\begin{aligned} &\frac{y^*(m+h) - y^*(m)}{h} \\ &\qquad = -\frac{\sqrt{1 - x^2(m+h)}}{h} \\ &\qquad = -\sqrt{(1 - x(m+h) \cdot \frac{1 + x(m+h)}{h^2}}. \end{aligned}$$

Now

$$\begin{aligned} x(m+h) &= x(m) + x'(m)h + \frac{x''(m)}{2}h^2 + \epsilon(h) \\ &= -1 + \frac{x''(m)}{2}h^2 + \epsilon(h) \end{aligned}$$

where $\dfrac{\epsilon(h)}{h^2} \to 0$ as $h \to 0$.

Hence,

$$\frac{x(m+h) + 1}{h^2} \to \frac{x''(m)^2}{2}$$

and so

$$\frac{y^*(m+h) - y^*(m)}{h} \to -\sqrt{x''(m)}.$$

as $h \to 0$. (since m is a minimum, $x''(m) > 0$.) Thus, y^* is differentiable and

$$y^{*\prime}(s) = \begin{cases} \dfrac{x(s)x'(s)}{\sqrt{1 - x(s)^2}} & \text{if } s \neq m \text{ ,} \\ -\sqrt{x''(m)} & \text{if } s = m \end{cases}$$

Since m is a minimum $x'(s) < 0$ for $s < m$. Also $x(s) \to 1$ as $s \to m$. By l' hospitals's rule

$$\begin{aligned} \lim_{s \to m^-} y^{*\prime} &= \lim_{s \to m^-} \frac{2xx'(x'^2 + xx'')}{2xx'} \\ &= -x''(m) \end{aligned}$$

and it follows that $y^{*\prime}$ is continuous, which means that $y \in C^1$.

[1] https://pennance.us Reference: Manfredo P. do Carmo. Differential Geometry of Curves and Surfaces.

3. Isoperimetric Inequaality.

   Let

$$\alpha(s) = (x(s), y(s)), \quad a \leq s \leq b$$

   be a simple, plane, closed, regular curve of length ℓ enclosing a region of area A. Then

$$A \leq \frac{l^2}{4\pi}.$$

   Proof.

   Without loss of generality, assume that g is parametrized by arclength. Let

$$\alpha^*(s) = (x(s), y^*(s))$$

   as in the lemma.

$$\begin{aligned} A &= \int_{\operatorname{Im}\alpha} x\,dy \\ &= \int_0^\ell x(s)y'(s)\,ds. \end{aligned}$$

   Since α^* is a circle, its area

$$\begin{aligned} A^* &= -\int_{\operatorname{Im}\alpha} y^* dx \\ &= -\int_0^\ell y^*(s)x'(s)\,ds. \\ &= 4\pi r^2. \end{aligned}$$

   Adding the two equations, and using the Cauchy Schwartz inequality yields

$$\begin{aligned} A + 4\pi r^2 &= \int_0^\ell xy' - y^*x'\,ds \\ &= \int_0^\ell (x, y^*) \cdot (y', -x')\,ds \\ &\leq \int_0^\ell \|(x, y^*)\| \|(y', x')\|\,ds \\ &= r\int_0^\ell \sqrt{x'^2 + y'^2}\,ds \\ &= r\int_0^\ell \|\alpha'\|\,ds \\ &= r\ell. \end{aligned}$$

   Thus,

$$\frac{A + \pi r^2}{2} \leq \frac{lr}{2}.$$

   Using the arithmetic-geometric mean inequality,

$$\sqrt{A} \cdot \sqrt{\pi r^2} \leq \frac{lr}{2}.$$

   and it follows that

$$A \leq \frac{\ell^2}{4\pi}. \tag{1}$$

4. Converse.

   Equality in (1) only if α is a circle.

   Proof.

   From the equality condition in the Cauchy Schwartz inequality, it must be that

$$(x, y^*) = c(y', -x').$$

   for some constant c. I.e.,

$$\begin{aligned} x &= cy' \\ y^* &= -cx' \end{aligned}$$

$$\begin{aligned} \frac{y'}{x} &= -\frac{x'}{y^*} \\ &= -\frac{x'}{\sqrt{1 - x^2}}. \end{aligned}$$

   In the case $0 \leq s \leq m$

$$\begin{aligned} y' &= -\frac{xx'}{\sqrt{1 - x^2}} \\ &= \left(\sqrt{1 - x^2}\right)'. \end{aligned}$$

   Since $x(0) = 1$ it must be that

$$y(s) = \sqrt{1 - x(s)^2},$$

   and so (x, y) lies on the upper semicircle. A similar argument shows that (x, y) lies on the lower semicircle when $m < s \leq 1$.