The Cylindrical Helix

Philip Pennance[1] – Version: February 2016.

1. Definition.

A regular curve γ with tangent vector field T and non zero curvature is a *cylindrical helix* if there exists a unit vector u and an angle θ such that

$$T(t) \cdot u = \cos\theta.$$

2. Claim.

The definition of cylindrical helix is invariant under reparametrization. I.e., if $\bar{\gamma}$ is a reparametrization then the following are equivalent:

(a) $T(t) \cdot u = \cos\theta$

(b) $\bar{T}(s) \cdot u = \cos\theta$.

Proof.

Since $\bar{\gamma}$ is a reparametrization, there exists a smooth function g such that

$$\bar{\gamma} = \gamma \circ g.$$

If $t = g(s)$, then

$$\begin{aligned} & T(t) \cdot u = \cos\theta \\ \iff\ & T(g(s)) \cdot u = \cos\theta \\ \iff\ & \bar{T}(s) = \cos\theta. \end{aligned}$$

3. Geometric Interpretation.

If γ is cylindrical helix γ parametrized by arc length the component of γ' in the u direction is a constant.

$$\frac{d}{ds}(\gamma' \cdot u) = \cos\theta.$$

This is not true for an arbitrary parameter since

$$\frac{d}{dt}(\gamma' \cdot u) = \cos\theta \frac{ds}{dt}.$$

4. Claim.

Let γ be a regular curve with non zero curvature κ . The following are equivalent

(a) γ is a cylindrical helix.

(b) $\dfrac{\tau}{\kappa}$ is constant.

Proof.

Let γ is a cylindrical helix. Then there exists a unit vector u and an angle θ such that

$$T(t) \cdot u = \cos\theta.$$

Differentiating and recalling that $T' = \kappa N$ reveals that $u \cdot N = 0$. Hence by orthonormal expansion in the fram $[T, N, B]$

$$u = (\cos\theta)T + (u\dot{B})B$$

Since $\|u\| = 1$

$$u \cdot B = \pm \sin\theta.$$

and so

$$u = (\cos\theta)T \pm (\sin\theta)B$$

$$\begin{aligned} u' &= (\cos\theta)T' \pm (\sin\theta)B' \\ &= (-\cos\theta)\kappa N \pm (\sin\theta)\tau N. \end{aligned}$$

But, $u' = 0$, and therefore,

$$\frac{\tau}{\kappa} = \pm \cot\theta,$$

a constant.

Remark, the choice of sign is determined by the direction of the helix.

Conversely, suppose that $\frac{\tau}{\kappa}$ is constant. We seek a unit vector u, and an angle θ such that $T \cdot u = \cos\theta$. Define θ by

$$\cot\theta = \frac{\tau}{\kappa},$$

[1] https://pennance.us   Reference: Manfredo P. do Carmo. Differential Geometry of Curves and Surfaces.

so that

$$\kappa \cos\theta - \tau \sin\theta = 0.$$

Let

$$u = (\cos\theta)T + (\sin\theta)B.$$

Then $\|u\| = 1$ and

$$\begin{aligned} u' &= (\cos\theta)T' + \sin(\theta)B' \\ &= (\kappa\cos\theta - \tau\sin\theta)N \\ &= 0. \end{aligned}$$

Hence u is a constant unit vector. Finally,

$$\begin{aligned} T \cdot u &= T \cdot [(\cos\theta)T + (\sin\theta)B] \\ &= \cos\theta. \end{aligned}$$

5. Example.

Show that

$$\gamma(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c}\right)$$

where $a > 0$ and $c = \sqrt{a^2 + b^2}$ is a unit speed cylindrical helix.

An easy computation reveals that $\|\gamma'\| = 1$ and that $\gamma' \cdot e_3 = \frac{b}{c}$. Now

$$\frac{b^2}{c^2} = \frac{b^2}{a^2 + b^2} < 1.$$

Hence, $\left|\frac{b}{c}\right| < 1$. It follows that there exists an angle θ such that $\cos\theta = \frac{b}{c}$. Therefore

$$\gamma' \cdot e_3 = \cos\theta$$

which means that γ is a cylindrical helix.

6. Example.

For the helix in example (5), find the Frenet apparatus $[T, N, B, \kappa, \tau]$.

Solution.

Differentiating twice:

$$\gamma'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right).$$

$$\gamma''(s) = \left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\cos\left(\frac{s}{c}\right), 0\right).$$

Since $\|\gamma'\|^2 = 1$,

$$T = \gamma' = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right).$$

$$\kappa = \|T'\| = \frac{a}{c^2}.$$

$$N = \frac{T'}{\kappa} = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right).$$

Notice that T is perpendicular to the z axis and points towards it. Next,

$$\begin{aligned} B &= T \times N \\ &= \left(\frac{b}{c}\sin\frac{s}{c}, \frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right). \end{aligned}$$

The torsion τ can be computed from $B' = -\tau N$. Since

$$B' = \left(\frac{b}{c^2}\cos\frac{s}{c}, -\frac{b}{c^2}\sin\frac{s}{c}, 0\right)$$

this yields $\tau = \frac{b}{c^2}$.