The Second Fundamental Form

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1. Claim: The map

$$dN_p : T_pS \to T_pS$$

is self adjoint.

Proof: It suffices to show that:

$$\langle dN_p(X_u), X_v\rangle = \langle X_u, dN_p(X_v)\rangle$$

i.e., that:

$$\langle \hat{N}_u, X_v\rangle = \langle X_u, \hat{N}_v\rangle$$

Differentiation of the relation

$$\langle \hat{N}, X_u\rangle = 0$$

with respect to v yields

$$\langle \hat{N}_v, X_u\rangle = -\langle N, X_{uv}\rangle$$

Similarly,

$$\langle \hat{N}_u, X_v\rangle = -\langle N, X_{vu}\rangle$$

The required result follows from the equality of the second partial derivatives.

2. Corollary. The map

$$II_p : T_pS \to \mathbb{R}$$

given by

$$II_p(v) = \langle -dN_p(v), v\rangle$$

is a quadratic form called the *second fundamental form*.

3. Corollary: T_pS has orthonormal basis $\{t_1, t_2\}$ of eigenvectors of $-dN_p$. The vectors $\{t_1, t_2\}$ are called the *principal directions*. The matrix of $-dN$ with respect to the basis $\{t_1, t_2\}$ is diagonal with diagonal elements equal to the eigenvalues. The eigenvalues $k_1 \le k_2$ are called the *principal curvatures* and satisfy

$$k_1 = \text{Min}\ \{II(v) : |v| = 1\}$$
$$k_2 = \text{Max}\ \{II(v) : |v| = 1\}$$

4. Definition: Let S be an orientable surface with unit normal vector field N and α is a unit speed curve in S so that $N \times \alpha'(t) \in T_{\alpha(t)}S$. Since

$$\langle \alpha', \alpha''\rangle = 0$$

the orthonormal expansion of α' with respect to the basis $[N, \alpha', N\times\alpha']$ yields

$$\alpha'' = \langle \alpha'', N\rangle N + \langle \alpha'', N \times \alpha'\rangle(N \times \alpha')$$

   (a) The projection of the curvature vector α'' onto T_pS is called the *intrinsic or geodesic curvature vector*.

   (b) The projection coefficient

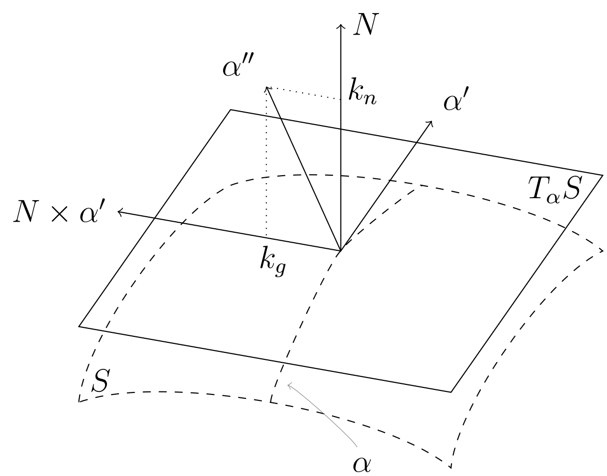
$$k_g := \langle \alpha'', N \times \alpha'\rangle$$

   is called the *geodesic curvature* or the *intrinsic curvature*

   (c) The projection coefficient

$$k_n := \langle \alpha'', N\rangle$$

   is called the *normal curvature* at $p = \alpha(t)$ in the direction $v = \alpha'(t)$.



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5. In this context the actual curvature vector α'' will be called the *extrinsic curvature vector* and $\kappa = |\alpha''|$ the *extrinsic curvature.*

6. Corollary:

$$\kappa^2 = k_n^2 + k_g^2$$

7. Geometrical Interpretation of the second fundamental form $I\!I_p(v)$:

$$I\!I_p(v) = k_n(v)$$

   Proof

$$\begin{aligned} k_n &= \langle \alpha'', N \circ \alpha \rangle \\ &= \langle \alpha', N \circ \alpha \rangle' - \langle (N \circ \alpha)', \alpha' \rangle \\ &= -\langle dN(\alpha'), \alpha' \rangle \\ &= I\!I_p(\alpha') \end{aligned}$$

8. Remark:

   The curvature of a surface contributes to the total turning of a curve within that surface. For example, viewed extrinsically, a great circle of the sphere possesses a constant non-zero curvature. However, from an intrinsic viewpoint, the great circle does not turn within the sphere. Rather, its curvature can be attributed entirely to that of the surface. It will be shown that the geodesic curvature k_g is a good measure of this intrinsic component of curvature. In the case of a great circle it turns out that $k_g = 0$.

9. Definition:

$$\begin{aligned} K &= \det(-dN) \\ &= k_1 k_2 \end{aligned}$$

   is called the *Gaussian curvature.*

$$\begin{aligned} H &= \tfrac{1}{2}\mathrm{Tr}(-dN) \\ &= \frac{k_1 + k_2}{2} \end{aligned}$$

   is called the *mean curvature* If α is a unit speed curve in S then

$$N \times \alpha'(t) \in T_{\alpha(t)}S$$

10. Claim [Euler]

    Let S be an oriented surface and $v \in T_pS$ a unit vector. Let k_1, k_2 be the principal curvatures at p. Then there exists θ such that

$$k_n(v) = k_1 \cos^2\theta + k_2 \sin^2\theta$$

    Proof: Let t_1, t_2 be the principal directions. Let

$$\begin{aligned} v &= \langle v, t_1 \rangle t_1 + \langle v, t_2 \rangle t_2 \\ &= \cos\theta \; t_1 + \sin\theta \; t_2 \end{aligned}$$

    where θ is the angle between v and t_1. Recall that

$$\begin{aligned} k_n(v) &= I\!I_p(v) \\ &= -\langle dN(v), v \rangle \end{aligned} \qquad (1)$$

    But

$$-dN(t_i) = k_i t_i, \quad i = 1, 2.$$

    Hence

$$-dN(v) = k_1 \cos\theta \; t_1 + k_2 \sin\theta \; t_2$$

    Finally, substitution for v and $dN(v)$ in (**??**) yields

$$k_n(v) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

11. Definition: Let S be an orientable surface with unit normal vector field N. Let α is a unit speed curve in S (so that $N \times \alpha'(t) \in T_{\alpha(t)}S$). Let π be the plane through p with normal $N \times \alpha'$ then the set $\pi \cap S$ is called the *normal section at p in the direction α'*

12. Definition: A curve $\alpha \in S$ will be called a *line of curvature* if $\alpha'(t)$ is a principle direction at $\alpha(t)$ for all t.