Fundamental Theorem of Curves - Uniqueness Part

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Claim.

Let α and $\bar{\alpha}$ be unit speed curves with the same curvature and torsion. Then there exists a rigid motion $F \in E^+(3)$ such that $\alpha = F \circ \bar{\alpha}$.

Proof.

Case 1. $\alpha(s_0) = \bar{\alpha}(s_0); \quad T(s_0) = \bar{T}(s_0); \quad N(s_0) = \bar{N}(s_0); \quad B(s_0) = \bar{B}(s_0)$.

Let

$$f(s) = \|T(s) - \bar{T}(s)\|^2 + \|N(s) - \bar{N}(s)\|^2 + \|B(s) - \bar{B}(s)\|^2.$$

Then,

$$\begin{aligned}
\tfrac{1}{2}f'(s) &= (T - \bar{T}) \cdot (T' - \bar{T}') + (N - \bar{N}) \cdot (N' - \bar{N}') + (B - \bar{B}) \cdot (B' - \bar{B}') \\
&= (T - \bar{T}) \cdot (-\kappa N' + \kappa \bar{N}) + (N - \bar{N}) \cdot (\kappa N - \tau B - \kappa \bar{N} + \tau \bar{B}) + (B - \bar{B}) \cdot (\tau N - \tau \bar{N}) \\
&= 0.
\end{aligned}$$

Hence, f is constant, in particular,

$$\begin{aligned}
f(s) &= f(s_0) \\
&= \|T(s_0) - \bar{T}(s_0)\|^2 + \|N(s_0) - \bar{N}(s_0)\|^2 + \|B(s_0) - \bar{B}(s_0)\|^2 \\
&= 0.
\end{aligned}$$

Since f is a sum of squares it must be that for all s, $T(s) = \bar{T}(s)$. It follows that $\alpha'(s) = \bar{\alpha}'(s)$. Since $\alpha(s_0) = \bar{\alpha}(s_0)$ it must be that $\alpha = \bar{\alpha}$.

Case 2. $\alpha(s_0) \neq \bar{\alpha}(s_0)$.

Since (T, N, B) and $(\bar{T}, \bar{N}, \bar{B})$ are orthonormal, right handed lists of vectors, there exists $M \in O^+(3)$ such that $\bar{T}(s_0) = MT(s_0)$, $\bar{N}(s_0) = MN(s_0)$, and $\bar{B}(s_0) = MB(s_0)$. Let

$$\bar{\bar{\alpha}} = M(\alpha(s) - \alpha(s_0)) + \bar{\alpha}(s_0).$$

Notice, $\bar{\bar{\alpha}}(s_0) = \bar{\alpha}(s_0)$ and $\bar{\bar{T}}(s_0) = \bar{\bar{\alpha}}'(s_0) = M\alpha'(s_0) = MT(s_0) = \bar{T}(s_0)$. Remembering that the dot product and norm are invariant under rotation $\bar{\bar{\kappa}}(s) = \|\bar{\bar{\alpha}}''(s)\| = \|M\alpha''(s)\| = \|T'(s)\| = \kappa(s) = \bar{\kappa}(s)$. Also

$$\bar{\bar{N}}(s_0) = \frac{\bar{\bar{T}}(s_0)}{\bar{\bar{\kappa}}(s_0)} = \frac{\bar{T}(s_0)}{\bar{\kappa}(s_0)} = \bar{N}(s_0),$$

$$\bar{\bar{B}}(s_0) = \bar{\bar{T}} \times \bar{\bar{N}}(s_0) = \bar{T} \times \bar{N}(s_0) = \bar{B}(s_0)$$

and

$$\bar{\bar{\tau}}(s) = -\bar{\bar{B}}'(s) \cdot \bar{\bar{N}}(s) = -B'(s) \cdot N(s) = \tau(s) = \bar{\tau}(s).$$

By case 1, $\bar{\alpha} = \bar{\bar{\alpha}} = F \circ \alpha$ where $F \in E^+(3)$ is given by

$$F(X) = M(X - \alpha(s_0)) + \bar{\alpha}(s_0).$$

[1] https://pennance.us Reference: Manfredo P. do Carmo. Differential Geometry of Curves and Surfaces.