Second Order Linear ODE's with Constant Coefficients: A Basis for the Kernel

Philip Pennance[1] – Updated: April 3, 2024

1. Introduction.

   To solve a linear equation, $Lx = 0$, differential or otherwise, means to find a basis for the kernel of L. Once a basis has been specified, every solution, is expressable as a inique linear combination of the elements of that basis, thus providing a finite specification of an infinite set. Moreover, changes of basis can simplify computations. For example $\{\cosh, \sinh\}$ and $\{e^t, e^{-t}\}$ are both bases for the solution space of $x'' - x = 0$.

2. Claim.

   Let $a, b \in \mathbb{R}$ and let $L : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}^0[\mathbb{R}]$ be the linear map

   $$L(\phi) = \phi'' + b\phi' + c\phi,$$

   then $\dim(\ker L) = 2$.

   Proof.

   By the existence and uniqueness theorem, there exist unique solutions, $\phi, \psi \in \ker(L)$, defined on all of $\mathbb{R}$ satisfying the initial conditions

   $$\begin{pmatrix} \phi(t_0) \\ \phi'(t_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   $$\begin{pmatrix} \psi(t_0) \\ \psi'(t_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

   We claim $\mathcal{B} = \{\phi, \psi\}$ is a basis for $\ker L$.

   To show that $\mathcal{B}$ is linearly independent, suppose that

   $$c_1\phi + c_2\psi = 0.$$

   Then

   $$\begin{aligned} c_1\phi(t_0) + c_2\psi(t_0) &= 0 \\ c_1\phi'(t_0) + c_2\psi'(t_0) &= 0 \end{aligned} \tag{1}$$

   Using the initial conditions, it follows that

   $$c_1 = c_2 = 0$$

   and so $\mathcal{B}$ is an independent set. To show that $\mathcal{B}$ is spanning, let

   $$\eta \in \ker L.$$

   We must show that η is a linear combination of ϕ and ψ.

   Define

   $$\xi = \eta(t_0)\phi + \eta'(t_0)\psi.$$

   Since $\ker(L)$ is a subspace, by closure we have that $\xi \in \ker L$. Substituting the initial conditions yields:

   $$\begin{aligned} \xi(t_0) &= \eta(t_0) \\ \xi'(t_0) &= \eta'(t_0). \end{aligned} \tag{2}$$

   I.e., ξ satisfies the same initial value problem as η. By uniqueness, $\eta = \xi$, proving that η is a linear combination of ϕ and ψ.

3. Claim.

   Let $L : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}^0[\mathbb{R}]$ be given by

   $$L\phi = \phi'' + b\phi' + c\phi \tag{3}$$

   and α_1, α_2 be the roots of the associated quadratic

   $$x^2 + bx + c = 0 \tag{4}$$

   (a) If α_1, α_2 are real and distinct then $\langle e^{\alpha_1 t}, e^{\alpha_2 t} \rangle$ is a basis for $\ker L$.

[1] https://pennance.us

(b) If $\alpha_1 = \alpha_2 = \alpha$ where α is real, then $\langle e^{\alpha t}, te^{\alpha t}\rangle$ is a basis for $\ker L$.

(c) If the quadratic has a conjugate pair of roots

$$\alpha_1 = \alpha + i\beta$$
$$\alpha_2 = \alpha - i\beta,$$

then $\langle e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t\rangle$ is a basis for $\ker L$.

Proof.

Since $\dim\ker L = 2$, it suffices to show, in each case, that the given functions are independent solutions of the homogeneous equation

$$x'' + bx' + cx = 0. \tag{5}$$

It is left to the exercises to verify that, in all cases, the Wronskian not does not vanish identically, (i.e., is not the zero function), thus proving linear independence. It only remains to verify that the functions are actually solutions!

In case (a),

$$\begin{aligned}&\left(e^{\alpha_1 t}\right)'' + b\left(e^{\alpha_1 t}\right)' + ce^{\alpha_1 t}\\ &= e^{\alpha_1 t}\left(\alpha_1^2 + b\alpha_1 + c\right)\\ &= 0.\end{aligned}$$

In case (b)

$$\begin{aligned}&\left(te^{\alpha t}\right)'' + b\left(te^{\alpha t}\right)' + cte^{\alpha t}\\ &= e^{\alpha t}\left(\alpha^2 t + 2\alpha + b\alpha t + b + ct\right)\\ &= e^{\alpha t}\left[t(\alpha^2 + b\alpha + c) + (2\alpha + b)\right]\\ &= 0\end{aligned}$$

(recall that in equation (4), $-b$ is equal to the sum of the roots. In this case $-b = 2\alpha$)

In case (c), let $z = \alpha + i\beta$ be one of the roots of the auxiliary quadratic, and notice that

$$e^{\alpha t}\cos\beta t = \operatorname{Re} e^{zt}$$
$$e^{\alpha t}\sin\beta t = \operatorname{Im} e^{zt}.$$

Since

$$\begin{aligned}&\left(e^{zt}\right)'' + b\left(e^{zt}\right)' + ce^{zt}\\ &= e^{zt}(z^2 + bz + c)\\ &= 0.\end{aligned}$$

It follows that both $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$ are solutions of the homogeneous equation.

4. Corollary.

   (a) If α_1, α_2 are real and distinct then (3) has solution

   $$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}.$$

   (b) If $\alpha_1 = \alpha_2 = \alpha$ where α is real, then equation (3) has general solution

   $$x(t) = e^{\alpha t}\left(A + Bt\right).$$

   (c) If the quadratic has a conjugate pair of roots

   $$\alpha_1 = \alpha + i\beta$$
   $$\alpha_2 = \alpha - i\beta$$

   then equation (3) has general solution

   $$e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right).$$

5. Example – Complex roots

   Solve
   $$4x'' - 4x' + 5x = 0.$$

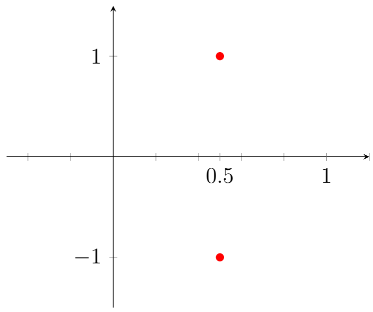
   Solution.

   The auxiliary equation

   $$4x^2 - 4x + 5 = 0$$

   has complex roots

   $$z = \frac{1}{2} \pm i.$$



A basis for the solution space is

$$\mathcal{B} = \{e^{t/2}\cos t, e^{t/2}\sin t\}.$$

Hence the general solution is

$$x = e^{t/2}(A\cos t + B\sin t).$$

6. Example.

The motion of an undamped spring of mass m and spring constant $k > 0$ satisfies $mx'' + kx = 0$. Let $\omega_0 = \sqrt{\frac{k}{m}}$. The auxiliary equation

$$x^2 + \omega_0^2 = 0$$

has roots $\pm i\omega_0$ and hence a basis for the solution space is

$$\mathcal{B} = \{\cos\omega_0 t, \sin\omega_0 t\}.$$

The general solution is therefore

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t.$$

7. Examples with Real Roots

ODE	Roots		Solution
$x'' + x' - 2x = 0$	-2	1	$Ae^{-t} + Be^{-2t}$
$x'' + x' = 0$	0	-1	$A + Be^{-t}$
$x'' - x' = 0$	0	1	$A + Be^{t}$
$x'' - x' - 6x = 0$	-2	3	$Ae^{-2t} + Be^{3t}$
$x'' - 3x' + 2x = 0$	1	2	$Ae^{t} + Be^{2t}$
$x'' = 0$	0	0	$A + Bt$
$x'' + 4x' + 4x = 0$	-2	-2	$(A + Bt)e^{-2t}$

8. Example.

Let $x(t)$ be the position of a ship at time t and x^* the position of a stationary iceberg. Suppose that after switching off the engines at time t_0, the motion of the ship is governed by the equation

$$mx'' + 2\gamma x' = 0$$

where $m, \gamma > 0$. If

$$x(t_0) = x_0 < x^* \qquad (6)$$
$$x'(t_0) = v_0 > 0 \qquad (7)$$

Show that the ship will never reach the iceberg providing

$$\gamma > \frac{mv_0}{2(x^* - x_0)}.$$

Solution.

Associated quadratic:

$$mx^2 + 2\gamma x = 0$$

has roots $x = 0, -2\gamma/m$. Hence

$$\begin{aligned} x(t) &= Ae^{0t} + Be^{-2\gamma t/m} \\ &= A + Be^{-2\gamma t/m} \end{aligned}$$

The initial conditions imply:

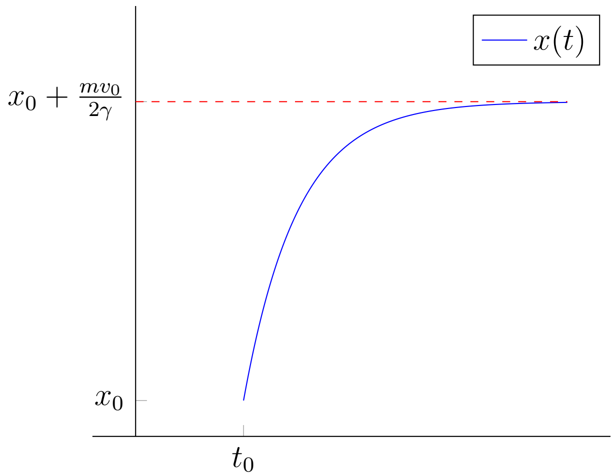
$$\begin{aligned} x_0 &= A + Be^{-2\gamma t_0/m} \\ v_0 &= -\frac{2\gamma}{m}Be^{-2\gamma t_0/m} \end{aligned}$$

It follows that:

$$\begin{aligned} B &= -\frac{mv_0}{2\gamma}e^{2\gamma t_0/m} \\ A &= x_0 + \frac{mv_0}{2\gamma} \end{aligned}$$

Hence the position of the ship at time t is:

$$x(t) = x_0 + \frac{mv_0}{2\gamma}\left[1 - e^{-2\gamma(t-t_0)/m}\right].$$



Notice that $x(t) \to x_0 + \dfrac{mv_0}{2\gamma}$ as $t \to \infty$. The ship will not reach the iceberg provided that:

$$
\begin{aligned}
x(t) < x^* &\Leftrightarrow x_0 + \frac{mv_0}{2\gamma} < x^* \\
&\Leftrightarrow \gamma > \frac{mv_0}{2(x^* - x_0)}
\end{aligned}
$$