First Order Linear Differential Equations

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1. Definition.

   A first order ODE is called *linear* if it has the form

   $$x' + p(t)x = q(t) \tag{1}$$

   where p and q be real valued functions which are continuous on an interval I. If $q = \underline{0}$ the zero function on I, (1) will be called the *homogeneous equation.*

2. Remark.

   Let
   $$L : \mathcal{C}^1[I] \to \mathcal{C}^0[I]$$
   be the diferential operator

   $$L(\phi) = (D + p(t))\phi,$$

   then (1) can be written

   $$Lx = q.$$

3. Corollary from Linear Algebra.

   Let S be the set of all solutions of (1). Then

   $$S = \phi_p + \ker L$$

   where ϕ_p is any solution of (1). ϕ_p is referred to as a *particular* solution.

   Recall,

   $$\ker L = \{\phi : L\phi = \underline{0}.\}$$

   I.e., the kernel is just the general solution of the homogeneous equation.

4. Example.

   Let q be a continuous function on $\mathbb{R}$. Find the kernel of the linear map

   $$D : \mathcal{C}^1[\mathbb{R}] \to \mathcal{C}^0[\mathbb{R}], \quad \phi \mapsto \phi'.$$

   and hence find the general solution of the trivial linear equation

   $$x' = q(t) \tag{2}$$

   Solution.

   For each $c \in \mathbb{R}$ define $\psi_c : \mathbb{R} \to \mathbb{R}$ by
   $$\psi_c(t) = c, \quad t \in \mathbb{R}.$$

   $$\begin{aligned} \ker D &= \{\phi : \phi' = 0, \quad t \in \mathbb{R}\} \\ &= \{\psi_c : c \in \mathbb{R}.\} \end{aligned}$$

   Thus, $\ker D$ is just the set of constant functions on $\mathbb{R}$. These form a vector subspace of $\mathcal{C}^1[\mathbb{R}]$ with basis $B = \{\psi_1\}$. By the fundamental theorem of calculus, any antiderivative Q of q is a particular solution. Hence, the solution set of (4) is

   $$S = \{\psi_c + Q : c \in \mathbb{R}\}.$$

   I.e., all solutions are of the form

   $$x(t) = c + \int^t q(\tau)d\tau.$$

   In particular, the solution of the initial value problem

   $$\begin{cases} x' = q(t), & (t, x) \in \mathbb{R}^2 \\ x(t_0) = x_0. \end{cases} \tag{3}$$

   is

   $$\phi(t) = x_0 + \int_{t_0}^t q(\tau)\, d\tau. \tag{4}$$

[1] https://pennance.us

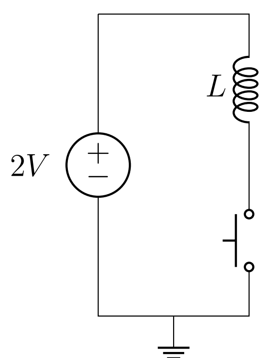
5. Example.

   The current i in an inductor (i.e,, a coil of wire) of inductance $L = 10mH$ is related to the voltage drop V_L across its terminals by the ODE

$$V_L = L\frac{di}{dt}$$

   The IVP

$$\begin{cases} i' = \frac{V_L}{L} \\ i(0) = 0. \end{cases} \tag{5}$$



   The equation represents the circuit above immediately after the switch is closed at $t = 0$. It follows immediately from (4) that

$$\begin{aligned} i(t) &= i_0 + \frac{V}{L}\int_0^t \, d\tau \\ &= \frac{2}{10(mH)}t \\ &= 200t \end{aligned}$$

   It is seen that the current ramps up at a rate of 200 amps per second.

6. Example

   Let $a \in \mathbb{R}$ be constant. The differential equation

$$x' = ax, \tag{6}$$

   can be written in the linear form

$$Lx = 0$$

   where L is the linear differential operator $D - a$ defined by

$$\begin{aligned} (D-a) : \mathcal{C}^1[R] &\to \mathcal{C}^0[R] \\ \phi &\mapsto \phi' - a\phi. \end{aligned}$$

   In this case the general solution is just the kernel of L. We claim that

$$\ker L = \{\phi_c : c \in \mathbb{R}\} \tag{7}$$

   where $\phi_c(t) = ce^{at}$. Substitution in the differential equation shows that

$$\{\phi_c : c \in \mathbb{R}\} \subseteq \ker L$$

   Conversely, if $\psi \in \ker L$, then

$$\begin{aligned} \frac{d}{dt}\psi(t)e^{-at} &= (\psi'(t) - a\psi(t))e^{-a(t)} \\ &= 0, \quad \forall t \in \mathbb{R}. \end{aligned}$$

   By the mean value theorem, it follows that

$$\psi(t)e^{-at} = c, \quad t \in I$$

   where c is a constant. I.e., ψce^{at}, and so

$$\ker L \subseteq \{\phi_c : c \in \mathbb{R}\}$$

   and (7) is proven.

7. Remarks.

   To understand how equation (6) arises naturally from processes of growth, consider a population which doubles every time unit. If $\phi(t)$ is the population at any time t and $x_0 = \phi(t_0)$ is the population at some particular time t_0, then

$$\phi(t_0 + n + 1) = 2\phi(t_0 + n), \quad n \in \mathbb{N}.$$

By induction, the population at time $t_0 + n$ satisfies

$$\phi(t_0 + n) = x_0 2^n, \quad n \in \mathbb{N}.$$

To pass from this discrete model to a continuous one, let $t = t_0 + n$ and assume n can take any real value. Then

$$\phi(t) = x_0 2^{t-t_0}.$$

Letting $a = \ln 2$, this equation can be rewritten

$$\phi(t) = x_0 e^{a(t-t_0)}, \quad t \in \mathbb{R} \tag{8}$$

A simple substitution now verifies that (8) is a solution of the initial value problem:

$$\begin{cases} x' = ax, & (t,x) \in \mathbb{R}^2 \\ x(t_0) = x_0. \end{cases} \tag{9}$$

It is also useful to think of the IVP

$$\begin{cases} x' = ax + q(t), & (t,x) \in \mathbb{R}^2 \\ x(t_0) = x_0. \end{cases} \tag{10}$$

as describing the growth of an investment at a fixed rate of interest[2] when an initial deposit x_0 is made at some time t_0. The function $q(t)$ in (10) captures the additional effects of deposits over time. Hence, we will refer to it as the *input function* and to $x(t)$ as the *response*. The following examples examine the special cases of constant, exponential, and sinusoidal inputs. A general formula for arbitrary input will also be obtained. In the lectures, the important special cases of delta function, and complex valued inputs will also be considered.

8. Example. (Constant input)

   Solve the IVP

$$\begin{cases} x' = ax + 1 & (t,x) \in D \\ x(0) = x_0. \end{cases} \tag{11}$$

Solution by undetermined coefficients.

We already know that the general solution of the homogeneous equation $x' = ax$ is just the kernel of the linear operator $L = D - 1$. Specifically,

$$\ker L = \{ce^a t : c \in \mathbb{R}\}.$$

Recall that the general solution of the non homogeneous equation (11), is obtained by adding a particular solution x_p to the kernel. The *method of undetermined coefficients,* which is known to work well for exponential, sinusoidal and polynomial inputs (but not much else), relies on making an intelligent guess at particular solution. In this case, since the input function $q(t)$ is a constant we try $x_p(t) = A$ where A is a constant. Substitution into the ODE shows that $A = -1/a$ and so

$$x_p(t) = -1/a$$

is a particular solution. Combining this with the solution of the homogeneous equation gives the general solution

$$x(t) = ce^{at} - 1/a. \tag{12}$$

[2] For $r\%$ interest, compounded continuously, $a = \frac{r}{100}$.

The initial condition $x(0) = x_0$ shows that $c = x_0 + 1/a$ and so

$$x(t) = x_0 e^a t + (1/a)(e^a t - 1).$$

Thus, the constant input $q(t) = 1$ has resulted in an increased response of $(1/a)(e^t - 1)$ with respect to the case of zero input.

9. Exercise.

   Solve the constant input case by separation of variables.

10. Example: Exponential Input.

    Solve the initial value problem

$$\begin{cases} x' = ax + e^{st}, & (t, x) \in \mathbb{R}^2 \\ x(0) = x_0. \end{cases} \tag{13}$$

Solution: By undetermined coefficients.

Since the input function e^{at} has the property that its derivatives are also exponential it is reasonable to seek particular solution of the form

$$x_p = \alpha e^{st}.$$

where α is a constant. Substituting x_p in (10) yields

$$\alpha = \frac{1}{s - a}$$

and so the general solution of the ODE is

$$x(t) = ce^{at} + \frac{1}{s-a} e^{st}, \quad c \in \mathbb{R}$$

From the initial condition we deduce that

$$c = x_0 - \frac{1}{s-a}$$

and it follows, in the case $s \neq a$, that

$$x(t) = x_0 e^{at} + \frac{e^{st} - e^{at}}{s - a}.$$

On the other hand, if $s = a$, the IVP becomes

$$\begin{cases} x' = ax + e^{st}, & (t, x) \in \mathbb{R}^2 \\ x(0) = x_0. \end{cases} \tag{14}$$

In this case the guess $x_p = \alpha e^{at}$ for a particular solution fails, since the source function e^{at} lies in the null space of the homogeneous equation. In this case, the method of undetermined coefficients suggests trying $x_p = \alpha t e^{at}$. Substituting this into the ODE yields $\alpha = 1$ and so (14) has solution

$$x(t) = x_0 e^{at} + te^{at}.$$

Notice, that this also follows directly from the solution in the case $s \neq a$ by taking the limit as s approaches a.

11. Example: Sinusoidal Input.

    Solve

$$x' = ax + \cos \omega t, \quad (t, x) \in \mathbb{R}^2 \tag{15}$$

Solution: By undetermined coefficients.

We seek a particular solutions of the form

$$x_p(t) = A \cos \omega t + B \sin \omega t.$$

Substituting in (15) yields (exercise)

$$A = \frac{-a}{\omega^2 + a^2}$$
$$B = \frac{\omega}{\omega^2 + a^2}.$$

and so a particular solution is

$$x_p(t) = \frac{-a}{\omega^2 + a^2}\cos\omega t + \frac{\omega}{\omega^2 + a^2}\sin\omega t.$$

Remark.

The solution can be written as a single sinusoid by using the identity

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\cos(\omega t - \alpha).$$

where $\alpha = \tan^{-1}(B/A)$.

12. Claim: The case of arbitrary input.

Let a be a constant and q a real valued function continuous on an interval I. The IVP

$$\begin{cases} x' = ax + q(t), & (t,x) \in \mathbb{R}^2 \\ x(t_0) = x_0. \end{cases} \tag{16}$$

has unique solution

$$x(t) = x_0 e^{a(t-t_0)} + \int_{t_0}^{t} e^{a(t-s)} q(s)\, ds, \quad t \in R. \tag{17}$$

Proof.

Multiplying (16) by the integrating factor e^{-at}, we have

$$\begin{aligned} & e^{-at}x'(t) - ae^{-at}x(t) = e^{-at}q(t) \\ \iff & \frac{d}{dt}x(t)e^{-at} = e^{-at}q(t) \\ \iff & x(t)e^{-at} = \int_{t_0}^{t} e^{-as} q(s)\, ds + c \\ \iff & x(t) = ce^{at} + e^{at}\int_{t_0}^{t} e^{-as} q(s) ds \end{aligned}$$

From initial condition, $c = x_0$.

Remark. Since ce^{at} is a solution of the homogeneous equation, it must be that (for any t_0)

$$x_p(t) = e^{at}\int_{t_0}^{t} e^{-as} q(s) ds$$

is a particular solution of the ODE (16).

13. Exercise.

Show that any other solution

$$\psi : I \to \mathbb{R},$$

of the ODE (16) is a restriction of that given by (17).

14. Claim.

Let K be the set of solutions of the ODE

$$x' = p(t))x \tag{18}$$

where p is continuous on an interval I and let P be any antiderivative of p. For each $c \in \mathbb{R}$, let $\phi : I \to \mathbb{R}$ be given by

$$\phi_c(t) = ce^{P(t)}, \quad t \in I.$$

Let $S = \{\phi : c \in \mathbb{R}\}$. Then

$$K = S.$$

Proof.

$$\begin{aligned} & \phi \in S \\ \iff & \phi(t) = ce^{P(t)}, \quad t \in I \\ \iff & \frac{d}{dt}\phi(t)e^{-P(t)} = 0, \quad t \in I \\ \iff & (\phi'(t) - p(t)\phi(t))e^{-P(t)} = 0, \quad t \in I \\ \iff & \phi'(t) - p(t)\phi(t) = 0, \quad t \in I \\ \iff & \phi \in K. \end{aligned}$$

15. Corollary

The kernel of the differential operator

$$\begin{aligned}(D - p(t)) : \mathcal{C}^1[I] &\to \mathcal{C}^0[I]\\ x &\mapsto x' - p(t)x\end{aligned}$$

is the one dimensional subspace of $\mathcal{C}^0[I]$ spanned by the function

$$\phi_1(t) = e^{P(t)}, \quad t \in I.$$

16. Example.

Find the general solution of the ODE

$$x' - (\sin t)x = 0$$

Solution.

The equation has the form (18) with $p(t) = \sin x$. An antiderivative is

$$P(t) = \int^t \sin s\, ds = -\cos t.$$

Hence the general solution

$$x(t) = ce^{-\cos t}, \quad c \in \mathbb{R}.$$

17. Let I be an interval and let $p, q \in \mathcal{C}^0[I]$. If P is an antiderivative of p, then the first order linear ODE

$$x' = p(t)x + q(t), \tag{19}$$

where p and q be real valued functions, continuous on an interval I, has particular solution

$$\phi_p(t) = e^{P(t)} \int^t e^{-P(s)}\, ds. \tag{20}$$

Proof.

Let ϕ_n be any solution of the corresponding homogeneous equation. The method of *variation of parameters* suggests seeking a particular solution of the form

$$\phi_p(t) = c(t)\phi_n(t), \quad t \in I. \tag{21}$$

Substituting in (19) and noting that $\phi_n'(t) = p(t)\phi_n(t)$ yields

$$c'(t)\phi_n(t) = q(t)$$

a solution of which is

$$c(t) = \int^t \frac{q(s)}{\phi_n(s)}\, ds.$$

Equation (20) now follows taking by $\phi_n(t) = e^{P(t)}$ in (21).

18. Corollary.

Under the preceeding hypotheses, the initial value problem

$$\begin{cases} x' = p(t)x + q(t) \\ x(t_0) = x_0. \end{cases} \tag{22}$$

has unique solution,

$$\phi(t) = x_0 e^{P(t) - P(t_0)} + e^{P(t)} \int_{t_0}^t e^{-P(s)}\, ds.$$

19. Example.

Solve

$$x' + (\sin t)x = \sin t$$

Solution.

To find the solution of the homogeneous equation, let

$$P(t) = \int^t (-\sin s\, ds = \cos t.$$

$$\begin{aligned} x_n \in \ker(D+\sin) &\iff x_n(t) = ce^{P(y)} \\ &\iff x_n(t) = ce^{\cos t}. \end{aligned}$$

A particular solution is

$$\begin{aligned} x_p(t) &= e^{P(s)} \int^t e^{-P(s)} q(s)\, ds \\ &= e^{\cos t} \int^t e^{-coss} \sin s\, ds \\ &= e^{\cos t} e^{-cost} \\ &= 1. \end{aligned}$$

The general solution of the inhimogeneous equations is

$$\begin{aligned} x(t) &= x_n(t) + x_p(t) \\ &= ce^{\cos t} + 1. \end{aligned}$$