Linear ODE's - Summary.

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1. Definition.

   Let V and W be vector spaces over $\mathbb{R}$. A function $f : V \to W$ is *linear* if for all $x, x' \in V$ and $c \in \mathbb{R}$

   (a) $f(x + x') = f(x) + f(x')$.

   (b) $f(cx) = cf(x)$.

2. Example.

   Let $L : \mathbb{R}^2 \to \mathbb{R}$ be linear. Show that there exist constants a and b such that

   $$L(x, y) = ax + by.$$

   Solution.

   Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and let e_1, e_2 be the standard basis for $\mathbb{R}^2$. Then

   $$\begin{aligned} L(\mathbf{x}) &= L(x_1e_1 + x_2e_2) \\ &= L(x_1e_1) + L(x_2e_2) \\ &= x_1L(e_1) + x_2L(e_2) \end{aligned}$$

   The result follows by taking $a = f(e_1)$ and $b = f(e_2.)$

3. Claim.

   Let I be an interval. The following are (infinite dimensional) real vector spaces under addition and scalar multiplication of functions:

   (a) $\mathcal{C}^0[I]$ - Functions continuous on I.

   (b) $\mathcal{C}^1[I]$ - Functions differentiable on I with continuous derivative.

   (c) $\mathcal{C}^n[I]$ - Functions n times differentiable on I with continuous n'th derivative.

   (d) $\mathcal{C}^\infty[I]$ - Functions which are infinitely differentiable on I.

4. Claim.

   The following maps are linear:

   (a) $D : \mathcal{C}^1[R] \to \mathcal{C}^0[I], \quad f \mapsto f'$.

   (b) $D^n : \mathcal{C}^n[R] \to \mathcal{C}^0[I], \quad f \mapsto f^{(n)}$

   Remark.

   The codomains of these maps reflect the fact that derivative of a differentiable function need not be differentiable.

5. Example.

   Show that $L : \mathcal{C}^2[R] \to \mathcal{C}^0[\mathbb{R}]$ given by by

   $$Lx = x'' + x \tag{1}$$

   is a linear map.

   Solution.

   Let f and g be twice continuously differentiable, then using the usual propeties of the derivative,

   $$\begin{aligned} L(f + g) &= (f + g)'' + (f + g) \\ &= (f'' + f) + (g'' + g) \\ &= L(f) + L(g) \end{aligned}$$

   and

   $$\begin{aligned} L(cf) &= (cf)'' + (cf) \\ &= c(f'' + f) \\ &= cL(f). \end{aligned}$$

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Hence L is linear.

6. Claim.

   Let $p, q \in \mathcal{C}^0[\mathbb{R}]$. Then the map
   $$L : \mathcal{C}^2[R] \to \mathcal{C}^1[R]$$
   given by
   $$L(\phi) = \phi'' + p(t)\phi + q(t).$$
   is linear.

7. Definition.

   Let V and W be linear spaces and
   $$L : V \to W$$
   a linear map. An expression of the form
   $$Lx = b,$$
   where $b \in W$, is called a *linear equation*. The set:
   $$S = \{x \in V : Lx = b\}$$
   is called the *solution space* or, in the case of a linear differential equation, the *general solution*.

8. Definition.

   Let $L : V \to W$ a linear map. The set
   $$\{x \in V : Lx = 0_W\}$$
   is called the *kernel* or the *null space* of L and is denoted $\ker L$. The equation
   $$Lx = 0_W$$
   is called the *homogeneous equation*. Thus the kernel of a linear map is just the solution of the homogeneous equation.

9. Claim.

   Let $L : V \to W$ a linear map. Then, $\ker L$ is a subspace of V.

   Proof.

   It suffices to note that $\ker L$ is non empty and closed under addition and scalar multiplication. (Exercise).

10. Example.

    Let
    $$L : \mathcal{C}^1[R] \to \mathcal{C}^0[R] : \phi \mapsto \phi'.$$
    Then
    $$\begin{aligned} \phi \in \ker L &\iff \phi' = 0 \\ &\iff \phi \text{ is constant.} \end{aligned}$$
    If for each $c \in \mathbb{R}$, we define
    $$\phi_c : R \to \mathbb{R} : \quad t \mapsto c, \quad \forall t \in \mathbb{R},$$
    then
    $$\begin{aligned} \ker L &= \{\phi_c : c \in \mathbb{R}\} \\ &= \langle \phi_1 \rangle. \end{aligned}$$
    where $\langle \phi_1 \rangle$ is the span of ϕ_1, the kernel is a one dimensional subspace of $\mathcal{C}^1[R]$.

11. Definition.

    Let $Lx = b$ be a linear equation. If $b \notin \operatorname{Im} L$, then there are clearly no solutions. If $b \in \operatorname{Im} L$ then there must exist at least one solution. Such a solution is called a *particular* solution.

12. Theorem.

Let $L : V \to W$ be a linear map. and x_p be a particular of a linear equation

$$Lx = b.$$

Then the set S of all solutions is the set

$$\begin{aligned} S &= x_p + \ker L \\ &= \{x_p + x_n : x_n \in \ker L\}. \end{aligned}$$

Proof.

Let $x \in S$. Then

$$Lx = b.$$

By definition of a particular solution, we also have

$$Lx_p = b.$$

It follows (by linearity of L) that

$$L(x - x_p) = O_W.$$

I.e.,

$$x \in x_p + \ker L.$$

Hence

$$S \subseteq x_p + \ker L.$$

Conversely, let

$$x \in x_p + \ker L.$$

Then there exists $x_n \in \ker L$ such that

$$x = x_p + x_n.$$

Then,

$$\begin{aligned} L(x) &= L(x_p + x_n) \\ &= L(x_p) + L(x_n) \\ &= b + O_w \\ &= b. \end{aligned}$$

Hence $x \in S$. It follows that

$$x_p + \ker L \subseteq S,$$

and so

$$S = x_p + \ker L.$$

13. Trivial Example.

Solve the differential equation

$$x' = 5.$$

using the method of theorem (12)

Solution.

Let

$$L : \mathcal{C}^1[R] \to \mathcal{C}^0[R] : \phi \mapsto \phi'$$

Recall,

$$\ker \phi = \{\phi_c : c \in \mathbb{R}\}$$

where ϕ_c is the constant function $\phi_c(t) = c$. Since $x_p(t) = 5t$ is a particular solution, the solution set

$$S = x_p + \ker L$$

consists of all functions of the form

$$\phi_c(t) = 5t + c, \quad c \in \mathbb{R}.$$

14. Solve the differential equation

$$x' - x = 3.$$

using the method of theorem (12).

Solution.

Let

$$L : \mathcal{C}^1[R] \to \mathcal{C}^0[R], \quad \phi \mapsto \phi' - \phi.$$

Recall, (see class notes) that

$$\ker \phi = \{\phi_c : c \in \mathbb{R}\},$$

where ϕ_c is the function $\phi_c(t) = ce^t$. By inspection $x_p(t) = 3, \quad t \in \mathbb{R}$ is a particular solution. The solution set $x_p + \ker L$ therefore, consists of all functions of the form

$$\phi_c(t) = 3 + ce^t, \quad c \in \mathbb{R}.$$

15. Example.

Solve the differential equation

$$x'' + x = 2t.$$

using the method of theorem (12).

Solution.

Let

$$L : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}^0[\mathbb{R}], \quad x \mapsto x'' + x.$$

As usual, $\ker L$ is just the set of solutions of the homogeneous differential equation

$$x'' + x = 0$$

Hence, as shown in the lectures,

$$\ker L = \{c_1 \sin + c_2 \cos : c_1, c_2 \in \mathbb{R}\}.$$

Since $(\sin, \cos)$ is a linearly independent list, this makes $\ker L$ a subspace of dimension 2.

It is easy to see that $x_p(t) = 2t$ is a particular solution. The general solution therefore is

$$\begin{aligned} S &= x_p + \ker L \\ &= \{2t + c_1 \sin t + c_2 \cos t : c_1, c_2 \in \mathbb{R}\}. \end{aligned}$$