Upper and Lower Fences

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1. Lemma.

   Let I be an open interval and

   $$\phi : I \to \mathbb{R}$$

   differentiable. Suppose $a \in I$. If

   $$\phi(a) = 0 \text{ and } \phi'(a) > 0$$

   then there exists $\delta > 0$ such that ϕ is positive in the interval $a < x < a + \delta$ and negative in $a - \delta < x < a$.

   Proof.

   Since ϕ is differentiable at a,

   $$\lim_{h \to 0^+} \frac{\phi(a+h) - \phi(a)}{h} = \phi'(a).$$

   Hence, for every $\epsilon > 0$, there exists $\delta > 0$ such that

   $$\left| \frac{\phi(a+h) - \phi(a)}{h} - \phi'(a) \right| < \epsilon$$

   whenever $0 < h < \delta$. Taking $\epsilon = \frac{\phi'(a)}{2}$ and noting that $\phi(a) = 0$, it follows that there exists $\delta > 0$ such that:

   $$0 < \frac{\phi'(a)}{2} < \frac{\phi(a+h)}{h}$$

   whenever $0 < h < \delta$. Since $h > 0$, it follows that $\phi(a+h) > 0$ whenever $0 < h < \delta$. I.e., ϕ is positive in a neighborhood immediately to the right of a. The proof that ϕ is negative immediately to the left of a is an easy exercise.

   Remark.

   If $\phi'(a) < 0$ a similar proof shows that ϕ is positive immediately to the left of a and negative immediately to the right.

2. Lemma.

   Let I be an open interval and $\phi : I \to \mathbb{R}$ differentiable. If $a, b \in I$ are consecutive zeros of ϕ with $a < b$ and $\phi'(a) > 0$, then $\phi'(b) \leq 0$. If $\phi'(a) < 0$ then $\phi'(b) \geq 0$

   Proof.

   Suppose to the contrary that $\phi'(b) > 0$. By the previous claim, ϕ will be positive to the immediate right of a and negative on the immediate left of b. By the Weierstrass intermediate value theorem, there exists $\xi \in (a, b)$ such that $\phi(\xi) = 0$, contradicting the fact that a and b are consecutive zeros of ϕ.

3. Corollary.

   Let $D \subseteq \mathbb{R}^2$ and $F : D \to \mathbb{R}$. If it happens that the t axis is one of the isoclines of the ordinary differential equation

   $$x' = F(t, x) \tag{1}$$

   and corresponding to a slope $m \neq 0$, then a solution of (1) which crosses the t axis cannot later recross.

4. Definition.

   Let $D \subseteq \mathbb{R}^2$ and $F : D \to \mathbb{R}$. Let $a, b \in \mathbb{R}$ with $-\infty \leq a < b \leq \infty$

   (a) A differentiable function

   $$l : (a, b) \to \mathbb{R}$$

   is a *lower fence* of F if

   $$l'(t) < F(t, l(t)), \quad a < t < b.$$

   (b) A differentiable function

   $$u : (a, b) \to \mathbb{R}$$

   is an *upper fence* of F if

   $$F(t, u(t)) < u'(t).$$

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5. Definition.

   A region

   $$\{(t,x) : l(t) < x < u(t), \quad a < t < b\}$$

   is called a *funnel of F* if l be a lower fence and u is an upper fence of F.

6. Claim.

   A solution of ϕ of (1) which crosses either a lower fence l or an upper fence u of F cannot recross.

   Proof.

   Let ϕ be a solution of (1) and let

   $$\alpha(t) = u(t) - \phi(t).$$

   Suppose, to the contrary that there exist consecutive crossings at $t = t_0$ and $t = t_1$ of ϕ and u then $\alpha(t_0) = \alpha(t_1) = 0$ and

   $$\begin{aligned} \alpha'(t) &= u'(t_0) - \phi'(t_0) \\ &= u'(t_0) - F(t_0, \phi(t_0)) \\ &> 0. \end{aligned}$$

   Similarly $\alpha'(t_1) > 0$. By lemma (2) this is impossible. Hence no more than one crossing is possible.

7. Exercise.

   Let $D = \mathbb{R}^2$ and let $F : D \to \mathbb{R}$ be given by $F(t,x) = t^2$. Consider the differential equation determined by F, namely, $x' = t^2$.

   (a) Use the isoclines (i.e. lines of constant slope) corresponding to slopes 0,1,4,9 to sketch the direction field for the differential equation.

   (b) Solve the differential equation and sketch a solution on the direction field.

8. Exercise.

   Describe the isoclines of each of the following ODE's. In each case sketch a direction field, and typical solution curves:

   (a) $x' = -t/x$

   (b) $x' = x^2 + t^2$

   (c) $x' = \dfrac{t}{x^2 + t^2}$

9. Exercise

   Let $D = \mathbb{R}^2$ and let $F : D \to \mathbb{R}$ be given by $F(t,x) = x^2 - t$. Consider the differential equation determined by F, namely, $x' = x^2 - t$.

   (a) Find and sketch the isoclines corresponding to slopes 0, and -1.

   (b) Show that $l(t) = -\sqrt{t}, \quad t > 0$ is a lower fence for F, i.e., show that $l'(t) < F(t, l(t)), \quad t > 0$. What does this tell you about solutions of the differential equation?

   (c) Show that $u(t) = -\sqrt{t-1}, \quad t > 5/4$ is an upper fence for F.

   (d) Show that $\lim_{t\to\infty}[u(t) - l(t)] = 0$.

10. Consider the ordinary differential equation:

    $$x' + x = t + 1 \tag{2}$$

    (a) Find the isoclines, draw a direction field and sketch typical solution curves of (2).

    (b) Prove that any solution which enters the region bounded by the isoclines corresponding to slopes $m = -1$ and $m = 2$ cannot escape this region.

    (c) Verify that $\phi(t) = t$ is a solution of (2). Find all solution of (2).