Elementary Symmetry Properties – Summary

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1. Claim.

   Let $v : U \subseteq \mathbb{R} \to \mathbb{R}$. Suppose that $\phi : I \to \mathbb{R}$ a solution of the autonomous equation $x' = v(x), \quad x \in U$.

   (a) If v is an odd function, then
   $$\eta : I \to \mathbb{R}$$
   given by
   $$\eta(t) = -\phi(t)$$
   is also a solution.

   (b) If v is an even function, then
   $$\eta : -I \ \to \mathbb{R}$$
   given by
   $$\eta(t) = -\phi(-t)$$
   is also a solution.

2. Exercise.

   Prove claims (1a) and (1b). Illustrate them by considering typical solutions of $x' = x$ and $x' = x^2$.

3. Claim.

   Let $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}$. Suppose that $\phi : I \to \mathbb{R}$ be a solution of the equation
   $$x' = F(t, x), \quad (t, x \in D).$$

   (a) If $F(-t, -x) = F(t, x)$, then $\eta : -I \to \mathbb{R}$, given by
   $$\eta(t) = -\phi(-t)$$
   is also a solution.

   (b) If
   $$F(-t, x) = -F(t, x),$$
   then $\eta : -I \to \mathbb{R}$ given by
   $$\eta(t) = \phi(-t)$$
   is also a solution.

   (c) If $F(t, -x) = -F(t, x)$, then $\eta : I \to \mathbb{R}$ given by
   $$\eta(t) = -\phi(t)$$
   is also a solution.

4. Exercise.

   Find all solutions of the following differential equations. Sketch the solutions and verify that your sketches are consistent with the symmetry conditions in claims (1a), (1b) and (3) .

   (a) $x' = 3t^2, \quad (t, x) \in \mathbb{R}^2$

   (b) $x' = x - t, \quad (t, x) \in \mathbb{R}^2$.

   (c) $x' = \dfrac{-t}{x}, \quad x \neq 0$.

   (d) $x' = 2xt, \quad (x, t) \in \mathbb{R}^2$.

   (e) $x' = -\dfrac{x}{\tanh t}, \quad t \neq 0$.

   (f) $x' = \dfrac{1}{2}(x^2 - 1), \quad x \in \mathbb{R}$.

5. Exercise.

   A point of mass m moving along a line experiences a force F which is a function of its position. Let $x(t)$ be the position of the point at time t. According to Newton's law (first written as a differential equation by Euler), the equation of motion is:
   $$mx'' = F(x). \tag{1}$$
   Show that solutions of (1) are invariant under time translations, i.e., show that if $x_1(t)$ is a solution of (1), then so is $x(t) = x_1(t + T)$ for any constant T.

[1] https://pennance.us