Time Invariant Linear Systems -Brief Review

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1. Introduction.

   Recall that if $a > 0$ is a constant and q a real valued function continuous on the interval $[0, \infty)$, then the first order linear differential equation

   $$x' = -ax + q(t) \tag{1}$$

   has steady state solution given by the convolution integral

   $$x_p(t) = \int_0^t e^{-a(t-\tau)} q(\tau)\, d\tau.$$

   In what follows, it is shown that this situation is far from accidental. Indeed, the steady state response of many mechanical, electrical and optical systems can be characterized by such a convolution.

2. Remark.

   Many physical systems can be characterized by the specification of their response to some stimulus $f(t)$. In this setting, it will be helpful to think of the inverse Fourier transform

   $$\begin{aligned} \hat{f}^{\check{}}(t) &= f(t) \\ &= \int \hat{f}(\nu) e^{2\pi i \nu t} dt \end{aligned}$$

   as a linear superposition of sinusuidal signals of frequency γ with amplitude specified by $\hat{f}(\nu)$. In the field of what is commonly known as Systems Theory, two problems arise naturally.

   (a) Modeling Problem.

   Given a function f is in some space, say $L^2(\mathbb{R})$, find a an operator L in this space so that $L[f(t)]$ is the response of the system at time t.

   (b) Realizability Problem.

   Given an operator L, does there exist a physical system which possesses the characteristics of the given operator?

   Both the modeling and realizability problems are made tractable by the imposition of constraints. The assumption of linearity is commonly made for physical systems subject to small perturbations when higher order terms can be ignored.

3. Definition.

   A system is *linear* if

   $$\begin{aligned} L[f(t) + g(t)] &= L[f(t)] + L[g(t)] \\ L[cf(t)] &= cL[f(t)] \end{aligned}$$

   for any inputs f, g and any constant c.

4. Definition.

   A system (described by an operator L) is called *time invariant* provided

   $$L[f] = g \implies L[f_\tau(t)] = g_\tau(t)$$

   for all τ. In this equation f_τ representes the translation of the function f. I.e.,

   $$f_\tau(t) = f(t - \tau).$$

5. Definition.

   The *impulse response* of a linear system is its response $h(t)$ to an impulse $\delta(t)$. I.e., $h(t) = L[\delta(t)]$

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6. Claim.

If a system is linear and time invariant with impulse response h , then

$$L[f] = h * f = \int_{\mathbb{R}} h(t-s) f(s)\, ds \qquad (2)$$

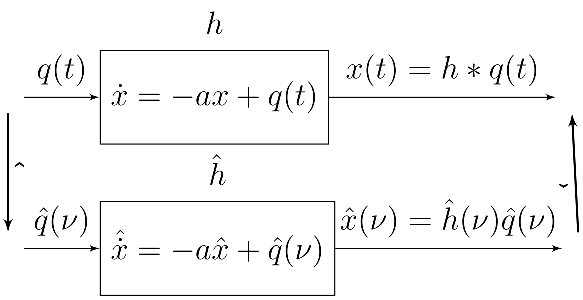
Proof.

$$f(t) = \int_{\mathbb{R}} \delta(s) f(t-s)\, ds = \int_{\mathbb{R}} \delta(t-s) f(s)\, ds$$

Hence

$$\begin{aligned} L[f(t)] &= L\left[\int_{\mathbb{R}} \delta(t-s) f(s)\, ds\right] \\ &= \int_{\mathbb{R}} L[\delta(t-s)] f(s)\, ds \\ &= \int_{\mathbb{R}} h(t-s) f(s)\, ds \\ &= (h * f)(t). \end{aligned}$$

Since Fourier (as well as Laplace and Z-transforms) map convolutions to products, the following diagram suggests a general solution technique using such transforms. (Details will be given in the course).



7. Remark.

The impulse response h can be found by subjecting a system to a suitable input, say, the delta function $\delta(t)$ or a Heaviside function. The following claim shows that the impulse response can be determined from the response to monochromatic input.

8. Claim.

The impulse response at a particular frequency represents the amplification and phase shift at that frequency.

Proof.

$$\begin{aligned} L\left[e^{i2\pi\nu t}\right] &= h * e^{i2\pi\nu t} \\ &= \int_{-\infty}^{\infty} h(\tau) e^{i2\pi\nu(t-\tau)} d\,\tau \\ &= e^{i2\pi\nu t} \int_{-\infty}^{\infty} h(\tau) e^{-i2\pi\nu\tau} d\,\tau \\ &= e^{i2\pi\nu t} \hat{h}(\nu). \end{aligned}$$

Remark.

It is seen from the preceeding result that the exponential functions behave like eigenfunctions of a linear, time-invariant operator.

9. Definition.

If h is the impulse response of a time invariant linear system, then

(a) $\hat{h}(\nu)$

(b) $|\hat{h}(\nu)|$

(c) $-\arg|\hat{h}(\nu)|$

(d) $-\log \hat{h}(\nu)$

are called respectively the *transfer function*, *gain*, *phase lag* and *attenuation* of the system.

10. Remark.

The term *filter* is often used in a loose fashion to refer to linear systems with gain $|\hat{h}(\nu)|$ negligible (in some sense) in some part or parts of the frequency axis.

11. Definition.

A filter L is called *distortionless* if there exist constants c, t_0 such that

$$L[f(t)] = cf(t-t_0), t \in \mathbb{R}$$

12. If L is a distortionless filter then the gain is a constant function, and the phase lag a linear function of γ.

$$\begin{aligned}\hat{h}(\nu) &= e^{-2\pi i\nu t}L[e^{2\pi i\nu t}] \\ &= ce^{-2\pi i\nu t_0}\end{aligned}$$

13. Definition.

    A time invariant linear system system is called *causal* or *non anticipative* if

$$f(\tau) = 0, \quad \tau < t \implies L[f(\tau)] = 0, \quad \tau < t.$$

14. Corollary.

    Since a causal system cannot respond to the future of the signal, the impulse response must satisfy

$$h(t) = 0, \quad t < 0,$$

    Moreover, since $h(t-\tau) = 0$ for $\tau > t$, the upper limit of integration ∞ in (2) can be repaced by t. I.e., the response of a causal system to an input $q(t)$ is the convolution integral

$$x(t) = \int_{-\infty}^{t} h(t-\tau)q(\tau)\,d\tau$$

15. In conclusion, the form of the steady state solution to the linear differential equation (1) can be viewed as the response function of a causal time invariant linear system. This fact generalizes to the steady state solutions of constant coefficient linear ODE's of higher order. Indeed, much more can be proven.

16. Theorem [Paley-Wiener]

    Necessary Condition for Causality. If $h \in L^2(\mathbb{R})$ is the impulse response of a causal system, then

    (a) $h(t) = 0$ for $t < 0$.

    (b) $\hat{h}(\nu)$ is a Hardy function.

    (c) $\displaystyle\int_{-\infty} \frac{\log|\hat{h}(\nu)|}{1+\nu^2}d\nu > -\infty.$

17. Paley and Weiner also solved the realizability problem for causal filters [1].

References

[1] Wiener's contributions to generalized harmonic analysis, prediction theory and filter theory P. Masani, Journal: Bull. Amer. Math. Soc. 72 (1966), 73-125.